

CAPITAL AND GROWTH

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ABSTRACT

This paper develops and discusses a neoclassical growth model with two inputs: physical capital stock and combined stock of human and intellectual capital. The production process is subject to diminishing returns to capital in perfect markets, in sharp contrast to new endogenous growth models that assume increasing returns to capital in imperfect markets. The model finds that a high saving rate raises both transitional and steady state growth rates of output through increases in physical, human, and intellectual investments that augment labor productivity—a key extension of the Solow (1956)-Swan (1956) growth model. Additionally, the paper derives an optimal rule for choosing the saving rate that maximizes consumer welfare. Implications for growth policies are drawn.

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I. INTRODUCTION

A basic proposition of the old neoclassical growth model [Solow (1956); Swan (1956) henceforth, S-S] is that the saving rate drives economic growth during the transition to the steady state, but owing to diminishing returns to capital, does not affect the steady state growth rate of per capita output, which is fixed by the rate of exogenous Harrod-neutral technical change.¹ On the latter, Solow (1991, p. 4) says: “Imagine an economy that has a constant, unchanging level of productivity. Then something happens—the invention of a computer, for instance—and productivity begins to rise. We know it will reach a new plateau and level off there. Then it will become constant again, higher than it was before but no longer changing. Such a process might take thirty years or even longer for a major invention. If you look at the annual growth rate, it will start at zero, build up to a positive value, perhaps quite suddenly, then start to fall back and reach zero again after thirty years have passed. I do not object to classifying this story as an interval of temporary growth. Such one-time gains in productivity are very valuable achievements.”

The S-S growth model motivates, and is nested in, this paper’s (henceforth, DV) model, summarized as follows.

The sources of economic growth in the S-S model are endogenous investments in physical capital during the transition to the steady state, and exogenous labor-augmenting technical change (exogenous investments in human and intellectual capital in the model of Section II) in the steady state.² The broader sources of growth in the DV model are endogenous and exogenous investments in all types of capital—physical, human, and intellectual—during the transition and in the steady-state. In Section II, the DV model shows that large exogenous investments in human and intellectual capital—as in the S-S model—and high saving rates raise the steady-state per capita output growth rate.³

The DV growth model captures the S-S model’s rich transitional dynamics—absent in new endogenous growth theories and models (see additional comments below). Increases in physical, human, and intellectual investments result in a burst of transitional output growth, overshooting the new and higher steady state per capita output growth rate.⁴ Another contribution of the model concerns the optimal

¹ Appendix A provides a review of the S-S growth model. Generally, output growth comes from two sources: capital growth and labor growth. Owing to diminishing returns to capital, ultimately the only other source of output growth is labor growth. In the S-S model, labor growth is equal to the sum of exogenous population growth and exogenous labor-augmenting technical change. Since the latter is exogenously fixed, no amount of saving can affect output growth in the steady state.

² For proof, see Appendix A, Equation (A12) for the transitional, and Equation (A14) for the steady-state, growth rate of per capita output. Although in the S-S model the saving rate does not affect the steady state “growth” of per capita output, a high saving rate nevertheless raises the steady state “level” of per capita output—see Equation (A17). In the absence of panel data with sufficiently long time series, econometric evidence pointing to positive growth effects of the saving rate is consistent with the transitional dynamics of the S-S model. See Knight *et al.* (1993).

³ Moreover, the DV model predicts that high rates of population growth and depreciation of physical capital lower the steady state per capita output growth rate—unlike in the S-S model where the steady state per capita output growth rate is determined exclusively by exogenous labor-augmenting technical change.

⁴ The latter steady state result is a major extension of the S-S model, owing to the DV model’s endogenous human and intellectual investments that are positive functions of the ratio of physical to human and intellectual capital (Section II).

choice of the saving rate, so that a unique value of the saving rate is obtained. The model employs the Golden Rule criterion suggested by Phelps (1966), based on maximization of real consumption.

The model's predictions are similar to those of endogenous growth models emphasizing R&D investments. What is different is that endogenous growth models assume increasing returns to capital— incompatible with balanced growth—and imperfect markets, while the DV model assumes diminishing returns to capital and perfect markets (standard neoclassical assumptions). The model's transitional dynamics is consistent with the empirical findings reported by Knight *et al.* (1993), using a novel panel data methodology. The empirical testing of the model's steady state predictions would have to await availability of very long-run data on a very large sample of countries.

The new endogenous growth theory [Romer (1986); Rebelo (1991), among others] questions the neoclassical S-S proposition that the saving rate does not affect the steady state output growth rate. By assuming constant or increasing returns to capital (broadly defined to include human capital), the new endogenous growth theory concludes that the economy's steady state output can grow as fast (or as slow) as the capital stock, and public policies with regard to saving affect steady state economic growth. In the AK model of Rebelo (1991), output Y is constant returns to capital K , implying that Y grows at the same rate as K , equal to sA (s multiplied by A), where s (larger than the saving rate in the S-S model by the amount of investment in human capital) is the fraction of income saved and invested, and A is a technological constant. In contrast to the S-S model, the AK model shows that both saving rate and technology determine the steady-state rate of output growth.⁵ Aghion and Howitt (1998) analyze a growth model with imperfect markets in the R&D sector characterized by Schumpeterian creative destruction. Along with Romer (1986), the knowledge-innovation-R&D production sector is subject to increasing returns to capital, so that economic growth does not fade away in the steady state.⁶

The present paper presents and discusses the DV model, addressing the following research questions. Retaining the neoclassical assumption of diminishing returns to capital, does the rate of saving, among other parameters, affect the steady state output growth rate? If the answer is yes, how does the former influence the latter? Using the Golden Rule criterion suggested by Phelps (1966), what is the optimal rule for choosing the saving rate that maximizes consumer welfare? Below is a summary of brief answers, explained and elaborated in the remainder of this paper.

First, the model finds that a high saving rate raises both the steady state and transitional per capita output growth rates through increases in physical, human, and intellectual investments that raise labor productivity (Section II). An optimal choice of the saving rate can be made using the Golden Rule criterion (Phelps,

⁵ The AK model has no transitional growth dynamics. Output growth always equals the steady state level, sA .

⁶ On increasing returns, Solow (1991, p. 12) comments: "As I have emphasized, the key assumptions all seem to require that some economic activity be exempt from diminishing returns. That is hard enough to test for a single industry or process, and even then might not settle the relevant question." Conlisk (1967) argues that increasing returns to capital yield explosive growth.

1966) or the Golden Utility criterion (Ramsey, 1928). The Phelps criterion is used in this paper (Section III).

Second, the DV model is neoclassical, in the tradition of old growth theory. Thus, it captures the S-S model's rich transitional dynamics—absent in new growth models of the AK variety. The policy implications of the S-S model are made wide-ranging by the DV model. Not only are saving policies effective in influencing the growth rate of per capita output at any point in time, but they can be used in “tilting” — to borrow Solow's word⁷—the steady state per capita output growth to a higher path. Increases in physical, human, and intellectual investments result in a burst of transitional output growth, overshooting the new and higher steady state growth rate—a key extension of the S-S model.

Third, to maximize social welfare, the net rate of return on capital should be greater than the sum of exogenous Harrod-neutral technical change and population growth, in order to compensate capital for magnified output growth generated by physical, human, and intellectual investments.⁸ Equivalently stated, capital's income share should exceed the saving rate to compensate capital for raising labor productivity and enhancing growth.⁹

The rest of the paper is organized as follows. Section II introduces the model, solves for and analyzes the uniqueness and stability of the steady state, and discusses its steady state and transitional growth dynamics. Section III derives the saving rate that maximizes consumer welfare using Phelps' (1966) Golden Rule. Section IV concludes with a summary and some implications for growth policy. Appendix A is a review of the S-S growth model. Appendix B provides the derivation and economic explanation of the DV growth model's key innovation regarding the dynamic equation for the stock of human and intellectual capital.

II. A NEOCLASSICAL MODEL OF CAPITAL AND GROWTH

Output Y is produced using as inputs physical capital K_p , human capital, and intellectual capital. For tractability, human capital and intellectual capital are combined in one capital input K_h . The stock of physical capital is the result of accumulated physical investment I . The stock of human and intellectual capital consists of accumulated human and intellectual investments V .¹⁰

Output I includes advanced capital goods (e.g., high-speed computers and modern industrial equipment). Output V includes education-training-experience of workers (gained at Harvard, MIT, Caltech, Silicon Valley and elsewhere), blueprints, methods, and processes to produce goods and services, including IT,

⁷ Solow (1991, p. 17).

⁸ In the S-S model, for maximum consumer welfare the net rate of return on capital should be equal to the sum of exogenous Harrod-neutral technical change and population growth. In Equation (21) of Section III, set $gw(k^*)=0$ and $gw'(k^*)=0$.

⁹ In the S-S model, capital's income share should equal the saving rate. In Equation (22) of Section III, set $\frac{I}{G(I,V)} = 1$, $gw(k^*) = 0$ and $gw'(k^*)=0$.

¹⁰ Solow's physical investment (Solow 1991, p. 15) is I as in the S-S model, and human-intellectual investments are captured by V (DV model's key innovation; for derivation, see Appendix B). The accumulated stocks of I and V are, respectively, K_p and K_h .

R&D, applied software development, Internet, Internet of Things, 5G technology¹¹, AI, Business Management Software and similar high-tech, intellectual activities.

The structural model of the paper is as follows, beginning with a unit-homogeneous neoclassical production function satisfying the Inada (1963) conditions (Cobb-Douglas).¹² Aggregate output Y is produced using two inputs K_p and K_h .

$$Y = K_p^\alpha K_h^{(1-\alpha)} \quad (1)$$

K_h is defined as

$$K_h = AN. \quad (2)$$

A constant fraction $(1-s)$ of Y is consumed

$$C = (1-s)Y \quad 0 < s < 1, \quad (3)$$

Y = aggregate output or income, K_p = stock of physical capital, K_h = stock of human and intellectual capital, $A = K_h^{-1}$ -augmenting productivity multiplier, N = working population¹³, C = consumption, a = output elasticity with respect to K_p , $(1-\alpha)$ = output elasticity with respect to K_h , s = saving rate and t = time (suppressed). Saved resources are used in the production of outputs I = physical investment (goods) and V = combined human and intellectual investments (services).

Substituting Equation (1) for Y ,

$$G(I, V) = sY = sK_p^\alpha K_h^{(1-\alpha)}. \quad (4)$$

$G(I, V)$ is assumed to be a unit-homogeneous joint index of I and V , with $\frac{\partial G}{\partial I} = G_1 > 0$ and $\frac{\partial G}{\partial V} = G_2 > 0$. Economic growth requires a high rate of saving s , so that more resources are made available to produce outputs I and V , in a proportion that depends on the ratio $\frac{K_p}{K_h}$.¹⁴

$$\frac{I}{V} = \phi\left(\frac{K_p}{K_h}\right) \quad \phi' < 0 \quad (5)$$

¹¹ See IHS Markit (2019) report on the importance of 5G technology to the global economy through 2035.

¹² With reference to the production function, $F(K_p, K_h) = K_h f(k)$, where K_p = physical capital, K_h = combined human and intellectual capital, and $k = \frac{K_p}{K_h}$, these conditions can be summarized as follows: $\lim_{\frac{\partial F}{\partial K_p} \rightarrow \infty} \text{as } K_p \rightarrow 0$; $\lim_{\frac{\partial F}{\partial K_p} \rightarrow 0} \text{as } K_p \rightarrow \infty$; $f(0) \geq 0$; $f'(k) > 0$, $f''(k) < 0$ for all $k > 0$. The Cobb-Douglas production function, Equation (1), satisfies these conditions.

¹³ Equation (2) is analogous to the S-S definition $L=AN$, where $L=K_h$, except that A in the S-S model is entirely exogenous and excludes saving-dependent investment in human and intellectual capital; see Equation (A5), Appendix A. Generally, the definition of L should be $L=APN$, where P is the labor participation rate, $0 < P \leq 1$. The working population is PN . When $P=1$, $L=AN$. Whatever P is, it is usually assumed as an exogenous constant, whose rate of change is zero. For an endogenous and variable P , see Villanueva (2020).

¹⁴ The saving parameter s needs a broader interpretation. It partly reflects the consumption-saving choice of society. It is also partly a production parameter since it determines factor intensity in the production of outputs I and V , relative to the production of C .

The assumption $\emptyset' < 0$ is reasonable. When the ratio of physical capital K_p to human and intellectual capital K_h rises (falls), the marginal product of K_p falls (rises) relative to the marginal product of K_h , and the economy produces less (more) I and more (less) V ; thus $\frac{I}{V}$ falls (rises).

Finally, k is the ratio of K_p to K_h :

$$k = \frac{K_p}{K_h} \quad (6)$$

The model consists of 6 equations in 6 variables (Y , K_p , K_h , I , V , and k). The variable C is determined by Equation (3) once the saving rate s is assigned an arbitrary value or an optimal value (Section III), or through the accounting definition $C=Y-G(I,V)$.

A. Reduced Model

The dynamic equations for the state variables K_p and K_h are given by:

$$\dot{K}_p = I - \delta K_p \quad (7)$$

$$\dot{K}_h = V + \lambda K_h + n K_h. \quad (8)$$

Equation (7) states that the increment in the physical capital stock \dot{K}_p equals gross fixed investment I less depreciation δK_p . Owing to the assumed unit-homogeneity of the output index $G(I,V)$, and using Equations (4) and (6), Equation (5) can be rewritten to show that gross fixed investment is equal to a fraction, $\frac{s}{G[1, \frac{1}{\emptyset(k)}]} = \frac{sI}{G(I,V)}$ of income Y ,

$$I = \frac{sY}{G[1, \frac{1}{\emptyset(k)}]}.^{15} \quad (9)$$

Substituting Equation (9) into Equation (7) and using $\frac{Y}{K_p} = k^{(\alpha-1)}$ [from Equations (1) and (6)],

$$\frac{\dot{K}_p}{K_p} = \frac{sk^{(\alpha-1)}}{G[1, \frac{1}{\emptyset(k)}} - \delta. \quad (10)$$

Equation (10) is the equation of motion for the physical capital stock.

Next is the derivation of the equation of motion for the combined stock of human and intellectual capital [Equation (12) below]. Equation (8) is the major innovation of the DV model and is the basis for Equation (12) using the relation $\frac{Y}{K_h} = k^\alpha$ [from Equations (1) and (6)]. Its derivation and economic explanation are detailed in Appendix B. Owing to the assumed unit-homogeneity of the output index $G(I,V)$, and using Equations (4) and (6), Equation (5) can be rewritten to

¹⁵ $G[1, \frac{1}{\emptyset(k)}] = \frac{G(I,V)}{I}$. Thus, $\frac{sY}{G[1, \frac{1}{\emptyset(k)}]} = \frac{sI}{G(I,V)} Y$, $0 < \frac{I}{G(I,V)} < 1$.

show that the sum of human and intellectual investments V is equal to a fraction $\frac{s}{G[\emptyset(k),1]} = \frac{sV}{G(I,V)}$ of income Y ,

$$V = \frac{sY}{G[\emptyset(k),1]} \tag{11}$$

Substituting Equation (11) into Equation (8) and using $\frac{Y}{K_h} = k^\alpha$ [from Equations (1) and (6)],

$$\frac{\dot{K}_h}{K_h} = \frac{sk^\alpha}{G[\emptyset(k),1]} + \lambda + n. \tag{12}$$

If $V=0$ (all saving is invested in physical capital), the model collapses to the S-S model, and the steady-state per capita output growth rate equals λ .¹⁷ Generally, however, saving is invested partly in physical capital I and partly in human and intellectual capital V , and both investments drive economic growth.¹⁸

Time differentiating Equation (6), and substituting Equations (10) and (12)

$$\frac{\dot{k}}{k} = \frac{\dot{K}_p}{K_p} - \frac{\dot{K}_h}{K_h} = \frac{sk^{(\alpha-1)}}{G[1, \frac{1}{\emptyset(k)}]} - \frac{sk^\alpha}{G[\emptyset(k),1]} - (\lambda + n + \delta). \tag{13}$$

At any point in time, the growth rate of output is given by time differentiating $\frac{Y}{K_h} = k^\alpha$ [from Equations (1) and (6)],

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}_h^*}{K_h} + \alpha \frac{\dot{k}}{k}. \tag{14}$$

$\frac{\dot{K}_h^*}{K_h}$ is given by Equation (12) evaluated at the steady state value of k at k^* , and $\frac{\dot{k}}{k}$ is given by Equation (13). Thus,

$$\frac{\dot{Y}}{Y} - n = \frac{sk^{*\alpha}}{G[\emptyset(k^*),1]} + \lambda + \alpha \frac{\dot{k}}{k}. \tag{15}$$

Equation (15) states that the transitional growth rate of per capita output $\frac{\dot{Y}}{Y} - n$ will be above (below) the steady state level $\frac{sk^{*\alpha}}{G[\emptyset(k^*),1]} + \lambda$ for a rising (falling) k , or whenever k is smaller (larger) than k^* .¹⁹ The phase diagram of the DV model is shown in Figure 1, containing plots of Equations (13) and (15). The vertical axis

¹⁶ $G[\emptyset(k), 1] = \frac{G(I,V)}{V}$. Thus, $\frac{sY}{G[\emptyset(k),1]} = \frac{sV}{G(I,V)}Y$, $0 < \frac{V}{G(I,V)} < 1$.

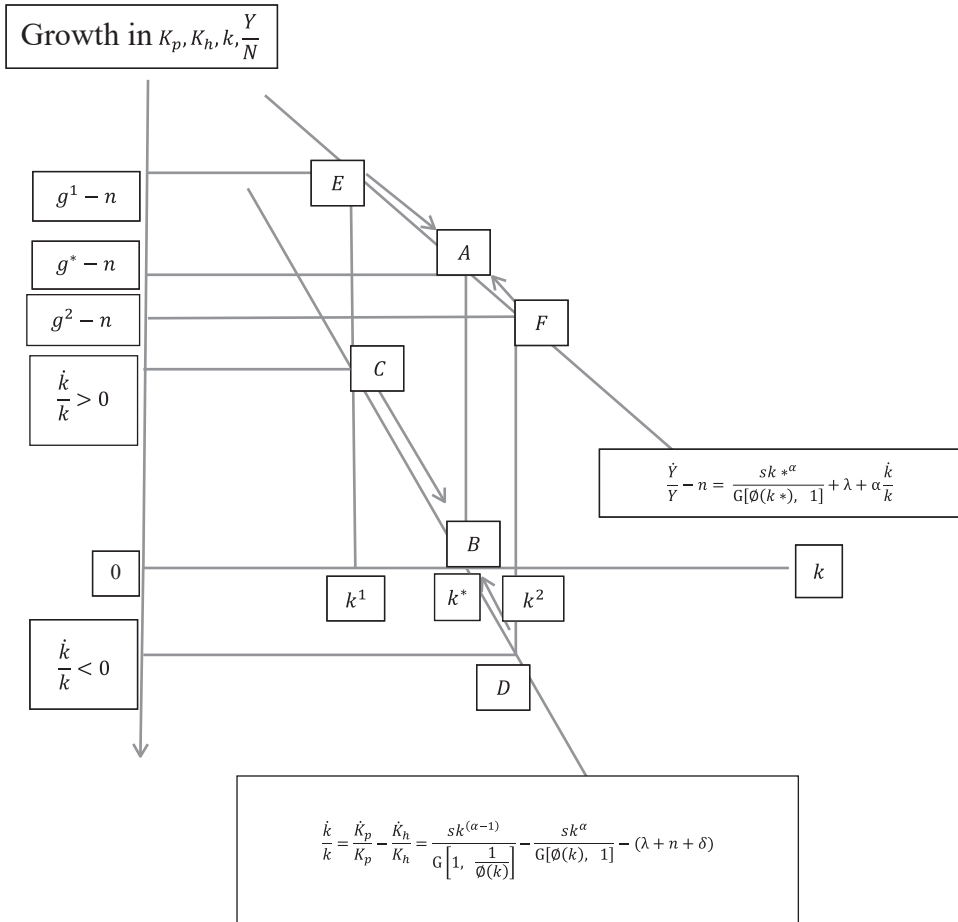
¹⁷ Set Equation (11) to zero, $V = \frac{sY}{G[\emptyset(k),1]} = 0$, so that Equation (12) becomes $\frac{\dot{K}_h^*}{K_h} = \frac{\dot{Y}^*}{Y} - n = \lambda$ (asterisk denotes steady state value). In the steady state, $k = \frac{K_p}{K_h}$ is constant at k^* ($\frac{\dot{k}}{k} = 0$), and $\frac{\dot{K}_p^*}{K_p} = \frac{\dot{K}_h^*}{K_h} = \frac{\dot{Y}^*}{Y}$. The latter equality to the steady state growth rate of output Y follows from the constant returns assumption on the production function, Equation (1).

¹⁸ Solow (1991) affirms that all physical, human and intellectual investments matter for growth.

¹⁹ It can also be seen from Equation (15) that the steady state growth rate of per capita output, evaluated at $k=k^*$, is equal to $\frac{\dot{Y}^*}{Y} - n = \frac{sk^{*\alpha}}{G[\emptyset(k^*),1]} + \lambda$, since the third term on the right-hand side (RHS) disappears when the economy reaches the steady state at $\frac{\dot{k}}{k} = 0$. See equilibrium points *A* and *B* in Figure 1.

measures the growth rates of K_p , K_h , k and $\frac{Y}{N}$ (transitional and steady state), and the horizontal axis measures $k = \frac{K_p}{K_h}$. Note that the $\frac{\dot{k}}{k}$ line is steeper than the $\frac{\dot{Y}}{Y} - n$ line, because in the latter, the slope of the first term on the RHS is positive (at a single value of k at k^*) and subtracts from the negative slope of the third term on the RHS (at all values of k).²⁰

Figure 1.
Equilibrium and Growth Dynamics



The steady state value k^* of k is obtained at the intersection of the $\frac{\dot{k}}{k}$ line with the k -axis at point B. At this steady-state equilibrium, balanced growth in physical capital $\frac{sk^{*(\alpha-1)}}{G[1, \frac{1}{\theta(k^*)}]}$ - δ , and in human-intellectual capital $\frac{sk^{*\alpha}}{G[\theta(k^*), 1]} + \lambda + n$, prevails at point A, and by the constant-returns assumption on the production function, the steady state growth rate of per capita output is $\frac{\dot{Y}}{Y} - n = g^* - n = \frac{sk^{*\alpha}}{[G[\theta(k^*), 1]} + \lambda$.

²⁰ The third term on the RHS is $\alpha \frac{\dot{k}}{k}$ and α is a fraction.

In Figure 1 it can be seen that at $k=k^*$, $\frac{\dot{k}}{k} = 0$ and the economy is on its balanced per capita output growth path at point $A(k^*, g^*-n)$. When k exceeds k^* , e.g., at k^2 , $\frac{\dot{k}}{k} < 0$ (point D), the per capita output growth rate $\frac{\dot{y}}{y} - n = g^2 - n$ (point F) is temporarily less than its steady state rate g^*-n . On the other hand, when k falls short of k^* , e.g., at k^1 , $\frac{\dot{k}}{k} > 0$ (point C), the per capita output growth rate $\frac{\dot{y}}{y} - n = g^1 - n$ (point E) is temporarily higher than the steady state rate g^*-n . The next paragraphs elaborate on the economics of the DV model's equilibrium behavior and growth dynamics both in the transition to and in the steady state.

B. Stability of Equilibrium and Growth Dynamics

In Figure 1 the equilibrium points A and B , characterized by the steady state k^* and steady state per capita output growth rate g^*-n , are not only unique but they are also globally stable.²¹ Consider any level of k to the left of k^* , such as k^1 , at which K_p grows faster than K_h ($\frac{\dot{k}}{k} > 0$ at point C) owing to a larger marginal product of K_p relative to that of K_h . As k rises from k^1 to k^* , the marginal returns on physical investment I fall and the marginal returns on human-intellectual investment V rise. Consequently, less I and more V are produced, and $\frac{I}{V}$ falls. Thus, \dot{K}_p slows, while \dot{K}_h accelerates, i.e., $\frac{\dot{k}}{k}$ becomes less and less positive until it falls to zero at the original equilibrium point B , or until k^1 has risen to k^* , traced by segment CB , to restore balanced growth. The opposite sequence of events unfolds for any initial value of k to the right of k^* , such as k^2 . At k^2 , K_h grows faster than K_p ($\frac{\dot{k}}{k} < 0$ at point D) owing to a smaller marginal product of K_p relative to that of K_h . As k^2 falls, the marginal returns on physical investment I rise and the marginal returns on human-intellectual investment V decline; thus, $\frac{I}{V}$ goes up, and \dot{K}_p accelerates while \dot{K}_h decelerates, i.e., $\frac{\dot{k}}{k}$ becomes less and less negative until it is zero at the original equilibrium point B (until k^2 has fallen to k^* , traced by segment DB).

The dynamics of the transitional per capita output growth rate is the following. At k^1 , output growth rate is g^1-n at point E , temporarily higher than the steady state growth rate g^*-n because the third term $\alpha \frac{\dot{k}}{k}$ on the RHS of the per capita output growth equation $\frac{\dot{y}}{y} - n$ in Figure 1 is positive ($\frac{\dot{k}}{k} > 0$) at point C . As k^1 rises towards k^* , the marginal returns on physical investment I decline and the marginal returns on human-intellectual investment V rise; $\frac{\dot{k}}{k}$ while still positive, decelerates, so that $\frac{\dot{y}}{y}$ falls from g^1-n to g^*-n . The adjustment process continues until the original equilibrium at point A is restored when k^1 settles at k^* , and the per capita output growth rate has fallen to the original equilibrium level g^*-n , traced by segment EA . The opposite sequence of events occurs when k is temporarily higher than k^* at k^2 . At k^2 , per capita output growth is g^2-n at point F , temporarily lower than

²¹ The Inada (1963) conditions—refer back to footnote 12— and the slope conditions [Equations (B10)–(B11)], Appendix B—ensure an intersection between the downward-sloping $\frac{\dot{k}}{k}$ line and the k -axis at some positive k , such as k^* at point $B(\frac{\dot{k}}{k} = 0)$ in Figure 1.

the steady-state growth rate g^*-n , because the third term $\alpha \frac{\dot{k}}{k}$ on the RHS of the output growth equation $\frac{\dot{Y}}{Y} - n$ at point D is negative ($\frac{\dot{k}}{k} < 0$). As k^2 falls towards k^* , the marginal returns on physical investment I go up and the marginal returns on human-intellectual investment V go down; $\frac{\dot{k}}{k}$ while still negative accelerates, so that $\frac{\dot{Y}}{Y} - n$ rises from g^2-n to g^*-n , traced by segment FA . The adjustment process continues until point A is reached, where k^2 stops falling and settles at k^* , and the per capita output growth rate has risen to the original equilibrium level g^*-n .

C. Comparative Dynamics

Recall that in the S-S model (Appendix A), with identical assumption on the production function (diminishing returns to inputs separately and constant returns jointly), the steady state growth rate of per capita output is invariant to changes in the saving rate (and in the other structural parameters except λ). For comparison, Table 1 and Figures 2-4 show the growth effects of the saving rate and the other structural parameters of the DV model.

Table 1.
Sensitivity of k^* and g^*-n to Parameter Changes

This table reports the sensitivity of k^* = equilibrium ratio of K_p to K_h and g^*-n = equilibrium growth rate of per capita GDP to all-else-equal increase in s = saving rate, λ = exogenous rate of human-intellectual investment, n = exogenous working population growth, and δ = rate of depreciation of the capital stock K_p , K_p = physical capital, K_h = combined human and intellectual capital.

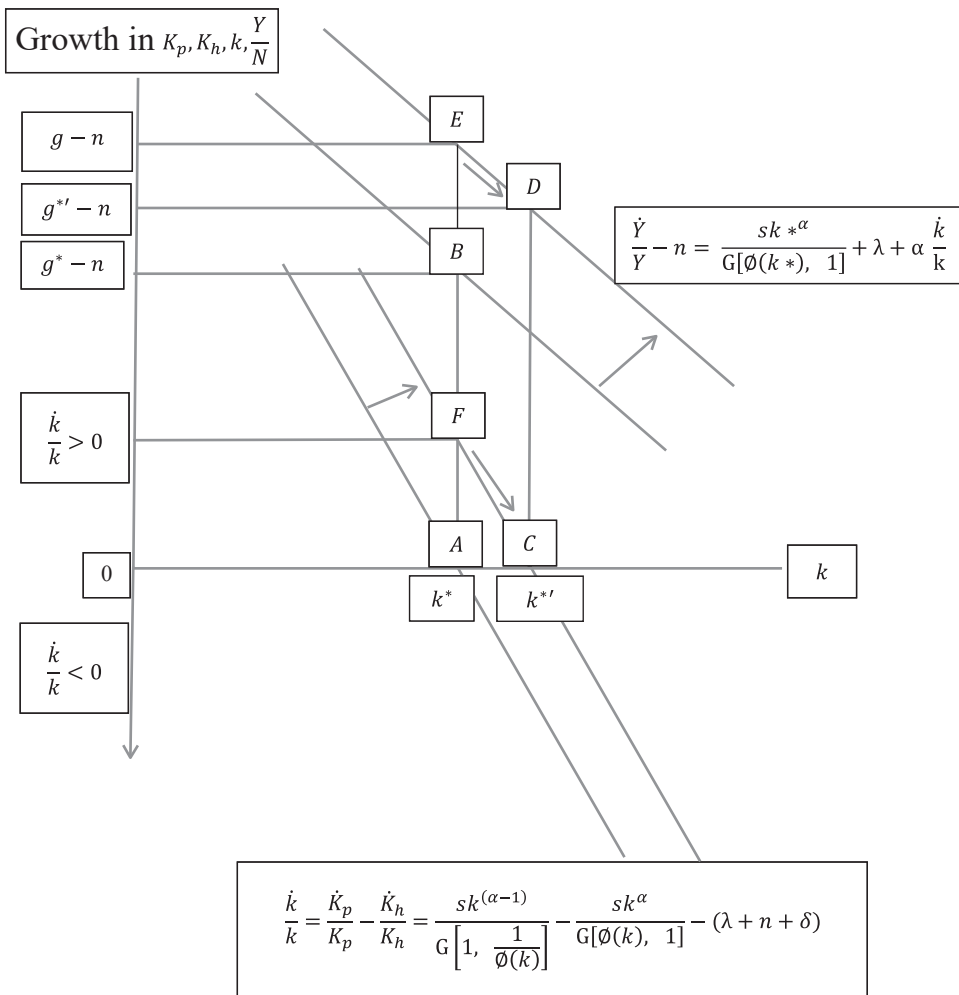
Description	s	λ	n	δ
Change in k^*	+	-	-	-
Change in g^*-n	+	+	-	-

Table 1 shows the sensitivity of k^* and g^*-n to parameter changes. The signs can be determined either algebraically or with reference to Figures 2-4. The solution to k^* is derived when Equation (13) is equated to zero, solving for k^* as an implicit function of s , λ , n and δ : $k^* = \vartheta(s, \lambda, n, \delta)$, with signs of the partial derivatives shown on the second row of Table 1. A higher k^* is associated with a higher saving rate, lower productivity of human-intellectual capital, lower population growth, and lower depreciation of physical capital. Evaluated at k^* , g^*-n is given by either Equation (10) or (12), either one as a function of k^* , s , λ , n and δ , or $g^*-n = \varphi(k^*; s, \lambda, n, \delta)$. The signs of g^*-n with respect to s , λ , n and δ can be derived by differentiation of $g^*-n = \varphi[\vartheta(s, \lambda, n, \delta); s, \lambda, n, \delta]$ with respect to each parameter. The signs are given by the third row of Table 1. A higher g^*-n is associated with a higher saving rate, higher productivity of human-intellectual capital, lower population growth, and lower depreciation of physical capital.

The signs and magnitudes of growth predictions in Table 1 can be empirically tested and estimated. All independent variables are observables, except for the exogenous K_h -augmenting productivity parameter λ , which can be impounded

in the constant term of the growth regressions.²² Panel data regressions would be appropriate to use in order to draw out the transitional growth effects of the independent variables. Knight *et al.* (1993) pioneered a panel data methodology that has been successfully used in estimating the transitional dynamics of growth models. As noted earlier, the DV—as well as the S-S—model’s transitional growth dynamics is consistent with the findings of Knight *et al.* (1993). However, averages of very long time series on a fairly large sample of countries are needed to test the DV model’s steady-state predictions shown in Table 1.

Figure 2.
Equilibrium and Growth Dynamics: Effects of an Increase in s



²² As noted earlier, K_h is analogous to effective labor L in the S-S model—see Appendix A, Equation (A2).

The signs in the second and third rows of Table 1 can also be determined through inspection of Figures 2-4. Figure 2 illustrates the transitional and steady-state growth effects of a higher saving rate s . The original equilibrium occurs at points $A(k^*, 0)$ and $B(k^*, g^*-n)$. Appendix B, Equations (B17)-(B19), Box 1B and related discussion demonstrate that the $\frac{\dot{k}}{k}$ equation shifts upward under the impact of a higher saving rate. The new $\frac{\dot{k}}{k}$ line intersects the k -axis at k^* (point C), higher than k^* . Meanwhile, the $\frac{\dot{y}}{y} - n$ line shifts upward because the RHS is larger (involving a higher s), resulting in a higher steady state per capita output growth rate g^*-n at point D.

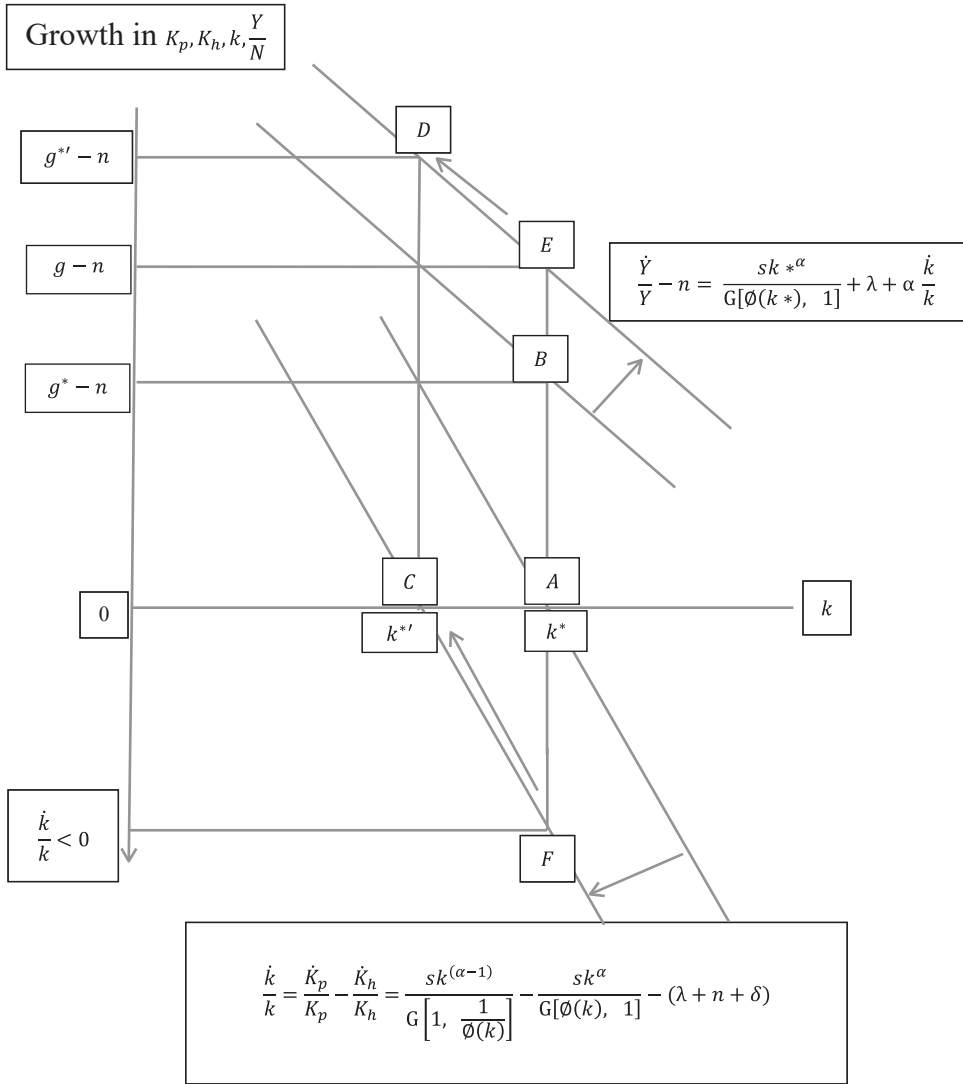
The economics of the transitional effects of a higher saving rate is the following. Immediately after the saving rate is raised through, for instance a higher fiscal surplus, at the initial k^* per capita output growth rate jumps to $g-n$ (segment k^*E), reflecting positive growth in k ($\frac{\dot{k}}{k} > 0$) at point F, feeding into a positive value for $\alpha \frac{\dot{k}}{k}$ in the $\frac{\dot{y}}{y} - n$ line. This outcome is a short-run expansionary overshooting of the next and higher steady state output growth g^*-n at point D. As k rises from k^* to k^* , the marginal returns on physical investment fall, and the marginal returns on human-intellectual investment rise. The third term on the RHS of the $\frac{\dot{y}}{y} - n$ line, $\alpha \frac{\dot{k}}{k}$ becomes less and less positive, leading to a deceleration of per capita output growth (adjustment from $g-n$ to g^*-n is traced by segment ED). This process continues until the next steady state at $D(k^*, g^*-n)$, when $\alpha \frac{\dot{k}}{k} = 0$, $(\frac{\dot{y}}{y})^* - n = g^* - n > g - n$, and $k^* > k^*$. The new result is a higher steady state growth rate of per capita output, compared with no change in the S-S model [see Equation (A14) and Figure 1A in Appendix A]. The reason can be seen by comparing Figure 2 with Figure 1A of the S-S model. The $\frac{\dot{y}}{y} - n$ line of the S-S model is $\frac{\dot{y}}{y} - n = \lambda + \alpha \frac{\dot{k}}{k}$, while the $\frac{\dot{y}}{y} - n$ line of the DV model is $\frac{\dot{y}}{y} - n = \frac{sk^*r^\alpha}{G[\emptyset(k^*), 1]} + \alpha \frac{\dot{k}}{k}$. When the saving rate goes up, the S-S steady state per capita output growth rate is unchanged at λ , whereas the DV model's steady state per capita output growth rate goes up because of a higher growth rate of K_h productivity $\frac{sk^*r^\alpha}{G[\emptyset(k^*), 1]}$ where $k^* > k^*$, resulting from higher saving-induced human and intellectual investments.²³

Next, consider the growth effects of changes in the other structural parameters λ , δ , and n . Figure 3 shows the growth dynamics following an increase in the exogenous K_h productivity rate λ . The starting equilibrium occurs at $A(k^*, 0)$ and $B(k^*, g^*-n)$. The higher K_h productivity λ shifts the $\frac{\dot{k}}{k}$ line downward, and the $\frac{\dot{y}}{y} - n$ line upward²⁴, leading to a new steady state equilibrium at points $C(k^*, 0), D(k^*, g^*-n)$, characterized by a higher steady state per capita output growth rate and a higher steady state ratio of human-intellectual capital to physical capital, reflecting higher endogenous human and intellectual investments. What happens to k and $g-n$ during the transition from initial steady state points A and B to new steady state points C and D ?

²³ Set $\alpha \frac{\dot{k}}{k} = 0$ in the expression for $\frac{\dot{y}}{y} - n$ in either S-S [Equation (A12), Appendix A] or DV [Equation (15)] model to obtain the steady-state per capita output growth rate.

²⁴ $\frac{\partial(\frac{\dot{y}}{y} - n)}{\partial \lambda} = (1 - \alpha) > 0$.

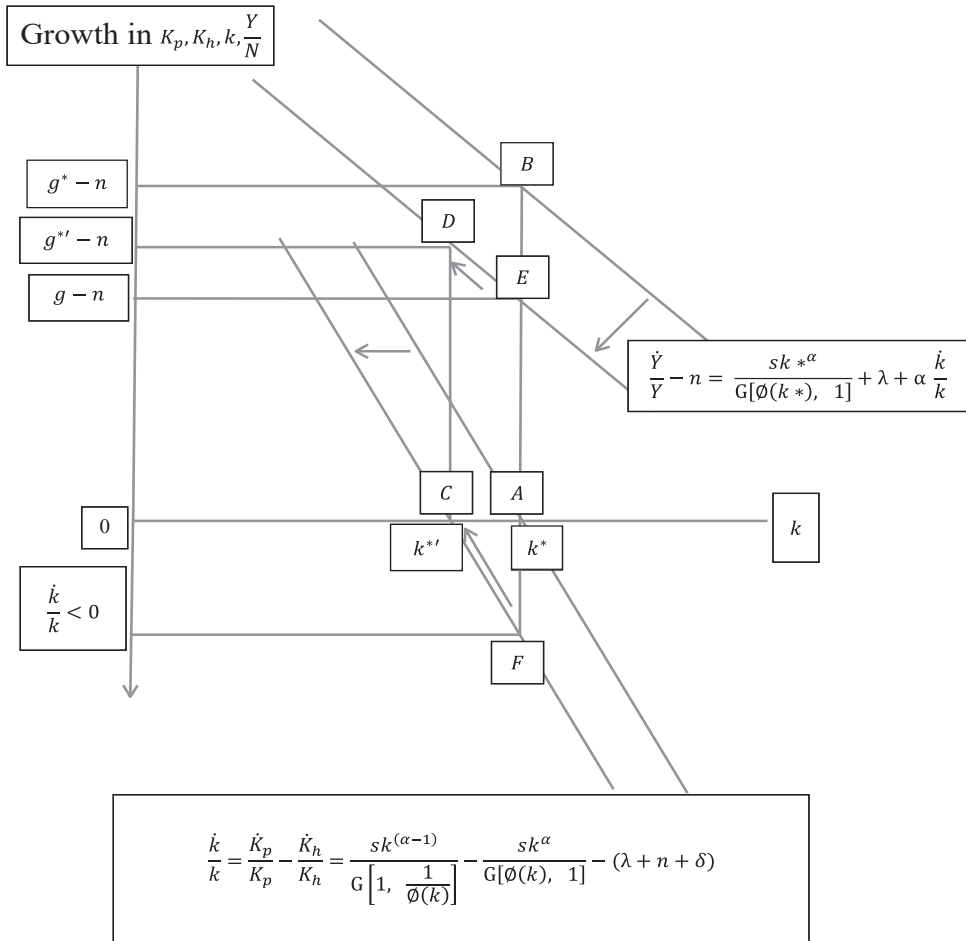
Figure 3.
Equilibrium and Growth Dynamics:
Effects of an Increase in λ



At the starting level k^* , and relative to its new steady state lower level k^{**} , physical capital's marginal product is less than that of human-intellectual capital following the rise in K_h productivity. Physical investment goes down, and human and intellectual investments go up. The proportionate rate of change in k turns negative at point F ($\frac{\dot{k}}{k} < 0$). As k^* contracts to its new steady state level k^{**} , the marginal returns on physical investment rise and the marginal returns on human and intellectual investments fall. The proportionate rate of change in k turns less and less negative until it becomes zero at point C (adjustment is traced by segment

FC in Figure 3). The growth effects of a higher exogenous productivity rate λ take place in two stages. At the initial k^* , the per capita output growth rate increases from g^*-n at B to $g-n$ at E. As k^* shrinks to k^{**} , higher returns on physical investment translate into higher income per unit of physical capital and higher saving-investment, i.e., higher warranted rate. When k^* has fallen to k^{**} , per capita output growth settles at the new steady state rate $g^{**}-n$ at D, still higher than the previous rate $g-n$.²⁵ Owing to endogenous human and intellectual investments, the higher productivity parameter λ leads to higher transitional growth, and eventually to higher steady state per capita output growth.²⁶

Figure 4.
Equilibrium and Growth Dynamics:
Effects of an Increase in n, δ



²⁵ The two-stage upward growth adjustment is traced by the path B-E-D.

²⁶ Recall the Solow quote on the temporary growth effect of the invention of a computer in the first paragraph of Section I. Instead of the λ -effect dissipating in the steady state as in the S-S model, the DV model yields a permanently higher steady-state warranted rate $g^{**}-n$, reflecting larger investments in physical capital because of higher returns on a smaller k^{**} .

Finally, Figure 4 illustrates the steady state and transitional growth effects of higher population growth and larger depreciation rate. The initial equilibrium occurs at points $A(k^*, 0)$ and $B(k^*, g^* - n)$. A higher rate of population growth lowers both $\frac{\dot{k}}{k}$ and $\frac{\dot{y}}{y} - n$ lines.²⁷ The new equilibrium occurs at points $C(k^*)$, and $D(k^*, g^* - n)$, characterized by lower steady state per capita output growth rate and lower steady state ratio of physical to human-intellectual capital.

The transitional growth dynamics is as follows. In Figure 4, a higher population growth rate n lowers both $\frac{\dot{k}}{k}$ and $\frac{\dot{y}}{y} - n$ lines (the latter through a drop in $\alpha \frac{\dot{k}}{k}$). Reading off the new and lower $\frac{\dot{k}}{k}$ line, at the starting k^* physical capital's marginal product is less than that of human-intellectual capital. Physical investment goes down, and human and intellectual investments go up. The proportionate rate of change in k turns negative at point F . As k^* contracts to its new steady state value k^* , the marginal returns on physical investment rise and those on human and intellectual investments fall, and the proportionate rate of change in k turns less and less negative until it is zero at point C (adjustment is traced by segment FC). Meanwhile, at the starting k^* , the per capita output growth rate, which starts at $g^* - n$ at B , drops sharply to $g - n$ at E . As k^* contracts to k^* higher returns on physical investment translate into higher income per unit of physical capital and thus, higher saving-investment, i.e., higher warranted rate. Growth recovers from $g - n$ to $g^* - n$. Note that this new steady state output growth is lower than the starting steady state growth rate $g^* - n$, a plausible prediction of the DV model.²⁸ The economics of the steady state and transitional growth dynamics of a higher rate of depreciation δ (Figure 4) follows similar arguments.

III. OPTIMAL SAVING RATE

In intensive form (as ratio to K_h) total output in the steady state is $y^* = k^{\alpha}$ [from Equations (1) and (6)]. If the level of y^* is considered a measure of the standard of living, and since $\frac{dy^*}{dk^*} = \alpha k^{*(\alpha-1)} > 0$, it is possible to raise living standards by increasing k^* . This can be done by adjusting the saving rate s . If consumption $c^* = \frac{c}{K_h}$ (or any monotonically increasing function of it) is taken as a measure of the social welfare of society, the saving rate that will maximize social welfare by maximizing steady state c^* can be determined. Phelps (1966) refers to this criterion as the Golden Rule of Accumulation.

Consumption is $c^* = \frac{C}{K_h} = \frac{Y}{K_h} - \frac{S}{K_h}$, where $S = I + V =$ saved output. $\frac{Y}{K_h} = y^* = k^{\alpha}$ and $S = I + V = \dot{K}_p + \delta K_p + \dot{K}_h - (\lambda + n)K_h$ [from Equations (7)-(8)]. Thus,

²⁷ $\frac{\dot{y}}{y} - n$ line shifts downward because the third term on the RHS ($\alpha \frac{\dot{k}}{k}$) goes down as n increases.

²⁸ The adjustment of per capita income growth is traced by a sharp fall from B to E , followed by a recovery from E to D . However, the new steady state per capita growth at point D remains lower than the initial one at point B —there is a permanent reduction in steady state per capita output growth resulting from higher population growth. Arrow (1962) discusses a learning-by-doing growth model with a prediction that higher population growth increases the per capita output growth rate. Available empirical evidence does not support such a hypothesis. Among other empirical studies, Conlisk (1967), Knight *et al.* (1993), and Villanueva (1994) find the opposite result, i.e., that higher population growth decreases the per capita output growth rate.

$$c^* = k^{*\alpha} - \left(\frac{\dot{K}_p}{K_p} \right)^* + \delta \Big) k^* - \left(\frac{\dot{K}_h}{K_h} \right)^* - (\lambda + n). \quad (16)$$

On the balanced growth path $\left(\frac{\dot{K}_p}{K_p} \right)^* = \left(\frac{\dot{K}_h}{K_h} \right)^*$, so that

$$c^* = k^{*\alpha} - \left(\frac{\dot{K}_h}{K_h} \right)^* (1 + k^*) - \delta k^* - (\lambda + n). \quad (17)$$

Now, evaluating Equation (12) at $k=k^*$,

$$\left(\frac{\dot{K}_h}{K_h} \right)^* = gw(k^*) + \lambda + n = \frac{sk^{*\alpha}}{G[\emptyset(k^*), 1]} + \lambda + n, \quad (18)$$

$$gw(k^*) = \frac{sk^{*\alpha}}{G[\emptyset(k^*), 1]}.$$

Substituting Equation (18) into Equation (17) yields,

$$c^* = k^{*\alpha} - [gw(k^*) + \lambda + n](1 + k^*) - \delta k^* - (\lambda + n). \quad (19)$$

Maximizing c^* with respect to s and equating to zero,

$$\partial c^* / \partial s = [\alpha k^{*(\alpha-1)} - \delta - gw(k^*) - \lambda - n - gw'(k^*)(1 + k^*)] \frac{\partial k^*}{\partial s} = 0 \quad (20)$$

Since $\frac{\partial k^*}{\partial s} > 0$,²⁹ the Golden Rule condition for maximum social welfare is,

$$\alpha k^{*(\alpha-1)} - \delta = gw(k^*) + \lambda + n + gw'(k^*)(1 + k^*) \quad (21)$$

Here, $gw(k^*) + \lambda + n + \delta + gw'(k^*)(1 + k^*)$ is the gross social marginal product of capital, inclusive of higher K_h productivity through human and intellectual investments induced by a higher k^* that is in turn triggered by a higher saving rate.³⁰

If there are no endogenous human and intellectual investments, the first-order condition reduces to $\alpha k^{*(\alpha-1)} - \delta = \lambda + n$, which is the S-S model's optimal rule for maximum consumer welfare.³¹ It is evident that the optimal net rate of return to capital should be higher than $\lambda + n$ when there are human and intellectual investments, i.e., $V^* > 0$ or $gw(k^*) > 0$, and $gw'(k^*) > 0$ or V^* responds positively to k^* ; see Equation (21).³² An alternative interpretation of the above Golden Rule can be

²⁹ For empirical support of this inequality, see Appendix B, Equations (B17)-(19), Box 1B, and related discussion.

³⁰ The first term on the RHS of Equation (21) $gw(k^*)$ is investment in new human and intellectual capital, and the fourth term $gw'(k^*)(1 + k^*)$ is the investment response to a higher k^* induced by a higher saving rate. The higher k^* lowers physical capital's marginal product and raises human-intellectual capital's marginal product, raising investments in the latter, and leading to higher steady state per capita output growth $g^* - n = \frac{\dot{K}_h}{K_h} = \frac{sk^{*\alpha}}{G[\emptyset(k^*), 1]} + \lambda$ [Equation (18)].

³¹ Set $gw(k^*) = 0, gw'(k^*) = 0$ in Equation (21).

³² Recall the discussion of Figure 2 on the growth effects of a higher k^* triggered by an increase in the saving rate. A higher k^* implies smaller returns on physical investment I and larger returns on human-intellectual investments V , resulting in less production of I and more of V .

given. A standard neoclassical (S-S) result is that the optimal saving rate s should be set equal to the income share of capital α —see discussion below. When there are endogenous human and intellectual investments V , the optimal saving rate should be set as a fraction of α , the fraction being equal to:

$$\frac{s}{\alpha} = \{[gw(k^*) + \lambda + n + \delta][\frac{I}{G(I,V)}]\} / \{[gw(k^*) + \lambda + n + \delta] + gw'(k^*)(1 + k^*)\} < 1 \quad (22)$$

The ratio $\frac{s}{\alpha}$ is less than unity because the ratio $0 < [gw(k^*) + \lambda + n + \delta] / \{[gw(k^*) + \lambda + n + \delta] + gw'(k^*)(1 + k^*)\} < 1$ and $0 < \frac{I}{G(I,V)} < 1$. Equation (18) and Equation (21) are used to derive Equation (22). In the S-S model, (a) $I = G(I,V) = sY$, i.e., all saving is invested in new physical capital, so that $\frac{I}{G(I,V)} = 1$ and (b) $gw(k^*) = 0, gw'(k^*) = 0$ (no saving-induced human and intellectual investments, or $V = 0$). Statements (a) and (b) together yield the S-S result, $\frac{s}{\alpha} = \frac{(\lambda + n + \delta)}{(\lambda + n + \delta)} = 1$.

The inequality Equation (22) can be restated as $\frac{\alpha}{s} > 1$. Equivalently stated, the income share of capital should exceed³³ the saving rate to compensate capital for the additional output generated by k^* -induced human and intellectual investments that augment labor productivity.

IV. SUMMARY AND CONCLUSION

This paper has presented and discussed a growth model consisting of two inputs: stock of physical capital; and stock of human and intellectual capital. The production process is subject to diminishing marginal returns to the two inputs separately and constant returns jointly in the context of perfect markets, in contrast to increasing returns to capital in imperfect markets assumed by endogenous growth models. Three major theoretical results are: (1) A higher saving rate raises both the steady state and transitional output growth rate through increases in physical, human, and intellectual investments that augment labor productivity; (2) The DV model captures the S-S model's rich and stable transitional dynamics—absent in endogenous growth models. Large physical, human, and intellectual investments result in a burst of transitional output growth, overshooting the new and higher steady state per capita output growth rate (the latter is a new result from the DV model, a major extension of the S-S model), owing to endogenous human and intellectual investments (positive function of the ratio of physical to human-intellectual capital k); and (3) To maximize social welfare, the net rate of return on capital should be greater than the sum of exogenous Harrod-neutral technical change and population growth in order to compensate capital for the positive growth generated by physical, human and intellectual investments. Equivalently stated, the income share of capital should exceed the saving rate in order to compensate capital for the output generated by a higher level of k , leading to higher K_h productivity. The implications for growth policy are straightforward.

³³ By the factor $\{[gw(k^*) + \lambda + n + \delta] + gw'(k^*)(1 + k^*)\} / \{[gw(k^*) + \lambda + n + \delta][\frac{I}{G(I,V)}]\}$.

Public policies that raise public sector saving for physical, human, and intellectual investments have magnified positive effects on the growth rate of per capita GDP. Strong incentives for private saving are essential for similar investments because of their positive growth effects in both the short-run (transition) and the long-run (steady state). The COVID-19 pandemic complicates the execution and implementation of public policies on saving and investments in physical, human, and intellectual capital. For countries with fiscal difficulties, such policies may have to await the resolution of the pandemic and restoration of sufficient fiscal space in order to create maximum growth effects. Meanwhile, the international lending community has an important role to play. Loans to purchase, distribute, and vaccinate the public should be made available to countries with sustainable foreign debt.³⁴

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³⁴ Extending the Ramsey (1928) optimal control procedure to an open economy and using a CRRA form of the utility function, Villanueva (2021) estimates sustainable gross foreign debt in range of 39-50 percent of GDP and domestic saving rates of 18-22 percent of GDP, depending on estimated elasticities of intertemporal substitution. These are feasible targets in developing and emerging market economies.

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APPENDIX A: REVIEW OF THE S-S GROWTH MODEL

The S-S model consists of the following relationships:

$$Y = K^\alpha L^{(1-\alpha)} \quad (\text{A1})$$

$$L = AN \quad (\text{A2})$$

$$I = S = sY \quad (\text{A3})$$

$$\dot{K} = I - \delta K \quad (\text{A4})$$

$$\frac{\dot{A}}{A} = \lambda \quad (\text{A5})$$

$$\frac{\dot{N}}{N} = n \quad (\text{A6})$$

$$k = \frac{K}{L} \quad (\text{A7})$$

$Y = \text{GDP}$, $K = \text{capital}$, $L = \text{effective labor}$, $A = \text{Harrod-neutral labor-augmenting productivity multiplier}$, $N = \text{population}$, $k = \text{capital intensity}$, $\alpha = \text{output elasticity with respect to capital}$, $1-\alpha = \text{output elasticity with respect to effective labor}$ ³⁵, $s = \text{gross fixed saving to income ratio}$, $\delta = \text{depreciation rate}$, $\lambda = \text{exogenous change in } A$ and $n = \text{population growth rate}$. A dot over a variable denotes time derivative, $\dot{K} = \frac{d(K)}{dt}$.

Y is produced according to a Cobb-Douglas production in Equation (A1), using K and L as inputs.³⁶ Equation (A2) defines L as the product AN .³⁷

Equation (A3) expresses the warranted rate in which investment is equal to saving, the latter being a fixed proportion s of income Y . Equations (A4)-(A6) are dynamic equations for the state variables K and L . Dividing Equation (A4) by K , using Equations (A1), (A3) and (A7),

$$\frac{\dot{K}}{K} = s \frac{Y}{K} - \delta = sk^{(\alpha-1)} - \delta. \quad (\text{A8})$$

Equation (A8) is termed the warranted rate. Time differentiating Equation (A2) and substituting Equations (A5) and (A6) yield,

$$\frac{\dot{L}}{L} = \lambda + n. \quad (\text{A9})$$

Equation (A9) is termed the natural rate.

³⁵ In Equation (A1), under assumed marginal factor productivity pricing and wage-price flexibility, the parameters α and $(1-\alpha)$ represent the income shares of capital and labor, respectively.

³⁶ Any unit-homogeneous function $Y=F(K,L)$ satisfying the Inada (1963) conditions will suffice. Given $F(K,L)=L^{\beta}f(k)$, where K is capital, L is effective labor, and k is the ratio of K to L , these conditions can be summarized as follows: $\lim_{k \rightarrow 0} \frac{\partial F}{\partial K} = \infty$ as $K \rightarrow 0$; $\lim_{k \rightarrow \infty} \frac{\partial F}{\partial K} = 0$ as $K \rightarrow \infty$; $f(0) \geq 0$; $f'(k) > 0$, $f''(k) < 0$ for all $k > 0$. The Cobb-Douglas production function [Equation (A1)] satisfies these conditions.

³⁷ Refer back to footnote 13.

Time differentiating Equation (A7) and substituting Equations (A8) and (A9) yield the proportionate change in the capital intensity k ,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} = \frac{\dot{I}}{L} = sk^{(\alpha-1)} - (\lambda + n + \delta). \quad (\text{A10})$$

From Equation (A1) and (A7), output in intensive form is:

$$\frac{Y}{L} = k^\alpha \quad (\text{A11})$$

Time differentiating Equation (A11) and substituting Equation (A9) yield the transitional growth rate of output,

$$\frac{\dot{Y}}{Y} = (\lambda + n) + \alpha \frac{\dot{k}}{k} \quad (\text{A12})$$

Substituting Equation (A10),

$$\frac{\dot{Y}}{Y} = \alpha[sk^{(\alpha-1)} - \delta] + (1 - \alpha)(\lambda + n). \quad (\text{A13})$$

Equation (A13) expresses the growth rate of Y during time t as the sum of capital and labor growth rates weighted by the income shares of capital α and labor $(1 - \alpha)$.³⁸

In the steady state, k is constant at k^* ($\frac{\dot{k}}{k} = 0$),³⁹ and by the constant-returns assumption,

$$\frac{\dot{K}^*}{K} = \frac{\dot{I}^*}{L} = \frac{\dot{Y}^*}{Y} = g^* = \lambda + n. \quad (\text{A14})$$

Equation (A14) is the steady state output growth rate, at which the warranted and natural rates are equal, and the economy is on a full-employment, balanced growth path.⁴⁰

Substituting Equation (A8) into Equation (A14), setting $k=k^*$,

$$sk^{*(\alpha-1)} = \lambda + n + \delta. \quad (\text{A15})$$

Solving for the equilibrium capital intensity,

$$k^* = \left[\frac{s}{(\lambda+n+\delta)} \right]^{\frac{1}{1-\alpha}}. \quad (\text{A16})$$

³⁸ Alternatively, Equation (A13) may be derived by time differentiating Equation (A1) and substituting Equations (A8) and (A9) into the result.

³⁹ The Inada (1963) conditions enumerated in footnote 36 ensure a unique and globally stable k^* .

⁴⁰ This is the S-S solution to the knife-edge problem posed by Harrod (1939)-Domar (1946), who employ a fixed output-capital ratio to conclude that balanced growth, macroeconomic stability, and full employment are not assured and may happen only by accident. S-S offers a variable output-capital ratio [Equation (A8)] as a solution. Another solution is found in Villanueva (2020) via a fully adjusting natural rate through endogenous labor participation, or through endogenous investments in human and intellectual capital [Equation (12) of the DV model], complementing a fully adjusting S-S warranted rate.

Equation (A16) states that the equilibrium capital intensity k^* is a positive function of the saving rate s , and a negative function of λ, n, δ . From Equations (A1), (A2), (A5), (A7), and (A16), steady state per capita output is given by:

$$\frac{Y^*}{N} = A(0)e^{\lambda t} \frac{s}{(\lambda+n+\delta)^{\frac{\alpha}{1-\alpha}}} \tag{A17}$$

In the S-S model, even though the steady state output growth rate is exogenously fixed by effective labor growth $\lambda+n$, independent of the saving rate s , the steady state level of per capita output $\frac{Y^*}{N}$ is a positive function of s .

Figure 1A.
Equilibrium and Growth Dynamics,
S-S Model:
Effects of an Increase in s

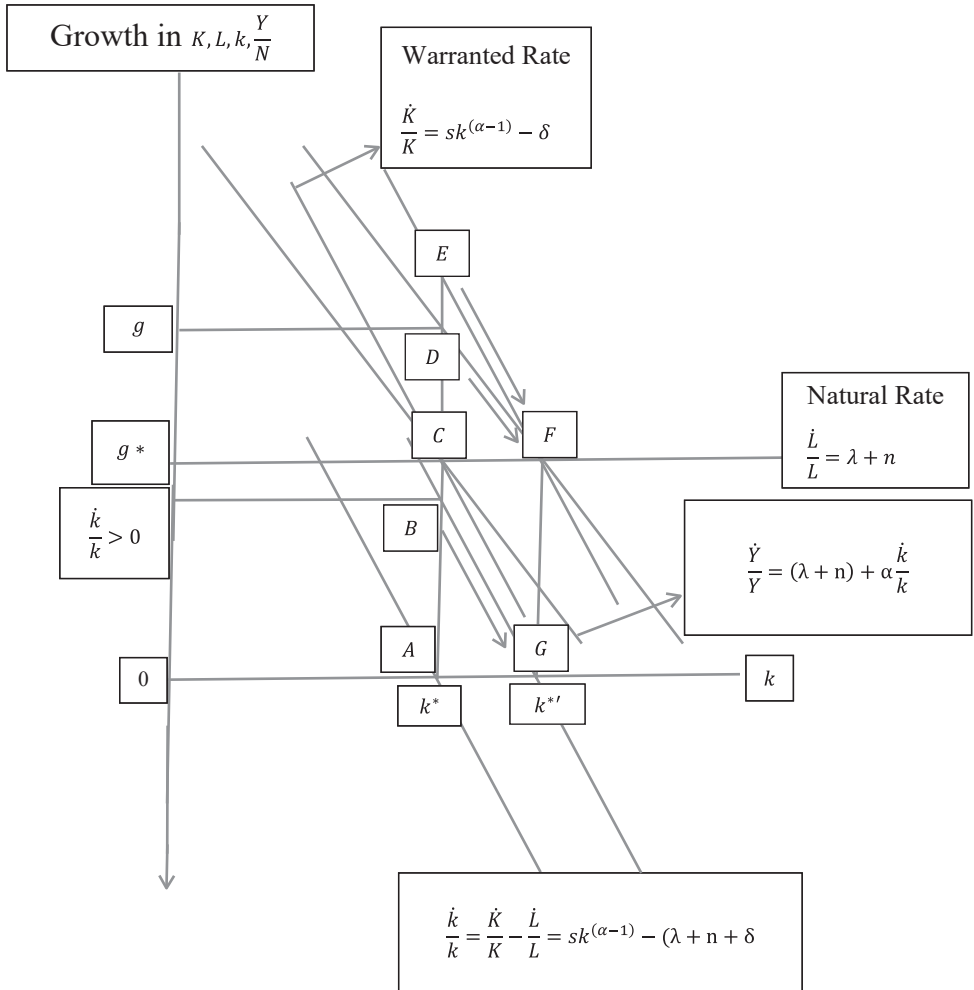


Figure 1A is the phase diagram showing the S-S model's equilibrium behavior and growth dynamics. It illustrates the steady state and transitional growth effects of an increase in the saving rate. The vertical axis graphs the rates of change in output $= \frac{\dot{Y}}{Y}$ [Equation (A12)], warranted rate $= \frac{\dot{K}}{K}$ [Equation (A8)], natural rate $= \frac{\dot{L}}{L}$ [Equation (A9)], and capital intensity $= \frac{\dot{k}}{k}$ [Equation (A10)]. The horizontal axis measures the level of capital intensity $= k$ [Equation (A7)].⁴¹

The steady state solution of the S-S model occurs at points $A(k^*, 0)$ and $C(k^*, g^*)$, at which the warranted and natural rates are equal (warranted rate line intersects natural rate line at point C), k^* is steady state capital intensity (the $\frac{\dot{k}}{k}$ line intersects the k -axis at point A), and $g^* = \lambda + n$ is steady-state output growth rate (reading off the $\frac{\dot{Y}}{Y}$ or $\frac{\dot{L}}{L}$ line). Equilibrium at point $A(k^*, 0)$ is unique and globally stable, ensured by the Inada (1963) conditions enumerated in footnote 36. Any capital intensity k different from k^* will bring k back to k^* because of the adjustments of the output-capital ratio and hence, of the saving-capital ratio $\frac{Y}{K}$, as capital's marginal and average products deviate from their equilibrium values at k^* . The warranted rate adjusts to the natural rate to bring balanced growth back to points $A(k^*, 0)$ and $C(k^*, g^*)$.

Assume an increase in the saving rate s , say through a higher fiscal surplus. The warranted rate, output growth, and capital intensity growth lines will shift upward to the right, while the natural rate remains stationary. The new steady state occurs at points $G(k'', 0)$ and $F(k'', g^*)$, characterized by higher equilibrium capital intensity with the same equilibrium output growth rate, because the natural rate is fixed at $g^* = \lambda + n$. More interesting is the transition to the new steady state. At the starting capital intensity k^* , a higher saving rate temporarily raises the warranted rate to point E (segment $k^* E$), which is larger than the natural rate (segment $k^* C$). Capital intensity growth turns positive (segment $k^* B$). Consequently, output growth goes up to g (segment $k^* D$) (reading off the new output growth line), that is temporarily higher than g^* . As capital intensity increases, the marginal returns on investment decline. The output-capital ratio falls, decreasing the saving-capital ratio and, hence, the warranted rate. This downward adjustment of the warranted rate (along segment EF) continues until it equals the natural rate at F , at which point the growth rate of output g reverts back to its original rate g^* (traced by the segment DF).⁴² Meanwhile, the growth rate of capital intensity turns less and less positive until it is zero at point G (traced by the segment BG), characterized by a higher level of capital intensity at k'' .

Figure 1A shows that, although the steady state output growth rate is fixed at $\lambda + n$, invariant with respect to the saving rate s , the output growth at any time is a function of s and all the other structural parameters of the model λ, δ, n . From Equations (A10) and (A13) and Figure 1A, an increase in the saving rate s will raise

⁴¹ The $\frac{\dot{K}}{K}$ line and $\frac{\dot{k}}{k}$ line are downward-sloping and parallel to each other because they have a common slope $s(\alpha-1)k^{(\alpha-2)}$. Both lines are steeper than the $\frac{\dot{Y}}{Y}$ line, whose slope is equal to $as(\alpha-1)k^{(\alpha-2)}$, where α is a positive fraction.

⁴² The output growth adjustment is traced by segment CDF. Solow's (1991) "temporary" growth is $g (>g^*)$, then back to $g^* = \lambda + n$.

the growth rate of capital intensity $\frac{\dot{k}}{k}$ and the transitional output growth rate $\frac{\dot{Y}}{Y}$. This rich dynamics is a major strength of the S-S model.⁴³

APPENDIX B: DERIVATION OF THE KEY INNOVATION OF DV MODEL

Take the time derivative of the main text Equation (2):

$$\dot{K}_h = \dot{A}N + A\dot{N}. \quad (\text{B1})$$

The increment in the human-intellectual capital stock \dot{K}_h is the sum of the increment in the K_h -augmenting productivity multiplier $\dot{A}N$ and the increment in the working population $A\dot{N}$. The increment in the K_h -augmenting productivity multiplier is

$$\dot{A}N = V + \lambda K_h \quad (\text{B2})$$

Incremental changes in K_h productivity $\dot{A}N$ result from human and intellectual investments (e.g., in education and R&D) captured by the variable V plus an exogenous component λK_h [analogous, but not identical, to Solow's λL ; see Appendix A, Equations (A2), (A5), (A6)].

The increment in the working population $A\dot{N}$ is given by

$$A\dot{N} = nK_h. \quad (\text{B3})$$

Equations (B1)-(B3) imply that the increment in human and intellectual capital \dot{K}_h is the sum of three elements: V , λK_h , and nK_h , explained below.

Repeating the main text Equation (11),

$$V = \frac{sY}{G[\theta(k), 1]}, \quad (\text{B4})$$

$$G[\theta(k), 1] = \frac{G(I, V)}{V}. \text{ Thus, } \frac{sY}{G[\theta(k), 1]} = \frac{sV}{G(I, V)} Y, \quad 0 < \frac{V}{G(I, V)} < 1.$$

Combining Equations (B1)-(B4),

$$\dot{K}_h = \frac{sY}{G[\theta(k), 1]} + \lambda K_h + nK_h. \quad (\text{B5})$$

Equation (B5) shows how V , the first element comprising the increment in the human and intellectual capital \dot{K}_h , is determined. V is equal to a fraction, $\frac{s}{G[\theta(k), 1]} = \frac{sV}{G(I, V)}$

⁴³ As noted earlier, this transitional dynamics is absent in the new endogenous growth models. Solow's thought experiment on the "temporary" growth effects of an increase in productivity (invention of a computer), quoted in the first paragraph of Section I, is an example. Section II discusses the DV model that preserves the S-S transitional dynamics. The new DV result points to a higher steady state per capita output growth rate in response to a higher saving rate (see Figure 2, Section II.C)-a generalization of the S-S model.

of income Y . The second element of \dot{K}_h is exogenous change in K_h productivity $\lambda K_{h'}$ roughly corresponding to the S-S model's rate of exogenous Harrod-neutral technical change λL . The third and final element of \dot{K}_h is an exogenous increase in the population of workers nK_h .

Using main text Equation (6), main text Equation (1) can be rewritten in intensive form as:

$$\frac{Y}{K_p} = k^{(\alpha-1)} \quad (\text{B6})$$

$$\frac{Y}{K_h} = k^\alpha \quad (\text{B7})$$

Repeating main text Equation (10),

$$\frac{\dot{K}_p}{K_p} = \frac{sk^{(\alpha-1)}}{G\left[1, \frac{1}{\vartheta(k)}\right]} - \delta. \quad (\text{B8})$$

Dividing Equation (B5) by $K_{h'}$ using Equation B(7),

$$\frac{\dot{K}_h}{K_h} = \frac{sk^\alpha}{G[\vartheta(k), 1]} + \lambda + n. \quad (\text{B9})$$

The slopes are given by:

$$\begin{aligned} \frac{d}{dk} \left(\frac{\dot{K}_p}{K_p} \right) &= G \left[1, \frac{1}{\vartheta(k)} \right]^{-1} [s(\alpha-1)k^{(\alpha-2)}] + \\ &sk^{(\alpha-1)}\vartheta'(k)G_2 \left[1, \frac{1}{\vartheta(k)} \right] \vartheta(k)^{-2} G \left[1, \frac{1}{\vartheta(k)} \right]^{-2} < 0 \end{aligned} \quad (\text{B10})$$

$$\frac{d}{dk} \left(\frac{\dot{K}_h}{K_h} \right) = G[\vartheta(k), 1]^{-1} s\alpha k^{(\alpha-1)} - sk^\alpha G_1[\vartheta(k), 1]\vartheta'(k)G[\vartheta(k), 1]^{-2} > 0 \quad (\text{B11})$$

The inequalities (B10) and (B11) imply that physical capital growth is a monotonically decreasing function of k , and human-intellectual capital growth is a monotonically increasing function of k .

Time differentiating the main text Equation (6) and substituting Equations (B8) and (B9) yield,

$$\frac{\dot{k}}{k} = \frac{\dot{K}_p}{K_p} - \frac{\dot{K}_h}{K_h} = \frac{sk^{(\alpha-1)}}{G\left[1, \frac{1}{\vartheta(k)}\right]} - \frac{sk^\alpha}{G[\vartheta(k), 1]} - (\lambda + n + \delta), \quad (\text{B12})$$

whose slope, given inequalities (B10)-(B11), is

$$\frac{d\left(\frac{\dot{k}}{k}\right)}{dk} = \frac{d}{dk} \left(\frac{\dot{K}_p}{K_p} \right) - \frac{d}{dk} \left(\frac{\dot{K}_h}{K_h} \right) < 0. \quad (\text{B13})$$

The steady-state k^* is the root of Equation (B12) equated to zero:

$$\frac{\dot{k}}{k} = \frac{sk^{*(\alpha-1)}}{G\left[1, \frac{1}{\vartheta(k^*)}\right]} - \frac{sk^{*\alpha}}{G[\vartheta(k^*), 1]} - (\lambda + n + \delta) = 0 \quad (\text{B14})$$

Evaluated at k^* , the steady state growth rate of per capita output g^* - n is given by either Equation (B8),

$$g^* - n = \frac{\dot{K}_p}{K_p} = \frac{sk^{*(\alpha-1)}}{G\left[1, \frac{1}{\theta(k^*)}\right]} - \delta - n, \quad (\text{B15})$$

or Equation (B9),

$$g^* - n = \frac{\dot{K}_h}{K_h} = \frac{sk^*}{G[\theta(k^*), 1]} + \lambda. \quad (\text{B16})$$

From Equations (B14)-(B16), a sensitivity matrix (signs indicated on the k^* and $g^* - n$ rows of Table 1 in the main text) can be derived either algebraically or with the help of Figures 2-4 in the main text—see discussion in Section II.C.

Figure 2 in the main text reproduces Figure 1 without reference to k^1 or k^2 on the k -axis, showing the steady state and transitional growth effects of an increase in the saving rate s . The initial steady state occurs at points $A(k^*, 0)$ and $B(k^*, g^* - n)$. When the saving rate increases, the first term on the RHS of the $\frac{\dot{Y}}{Y} - n$ line is larger, shifting this line upward. Whether the $\frac{\dot{k}}{k}$ line shifts upward as shown in Figure 2 is not obvious because the saving rate s enters in the first and second terms on the RHS of the $\frac{\dot{k}}{k}$ line symmetrically with opposite signs. For $\frac{\partial k^*}{\partial s} > 0$, i.e., for k^* at point A to increase to k^{**} at point C , it must be demonstrated that the $\frac{\dot{k}}{k}$ line shifts upward.

For the $\frac{\dot{k}}{k}$ line to shift upward, it must be shown that the upward shift of the $\frac{\dot{K}_p}{K_p}$ line is larger than the upward shift of the $\frac{\dot{K}_h}{K_h}$ line. Owing to the assumed unit-homogeneity of the joint output index $G(I, V)$, and noting that $\frac{Y}{K_p} = k^{(\alpha-1)}$ and $\frac{Y}{K_h} = k^\alpha$, Equations (B8)-(B9) can be rewritten as,

$$\frac{\dot{K}_p}{K_p} = \frac{sk^{(\alpha-1)}}{G\left[1, \frac{1}{\theta(k)}\right]} - \delta = \frac{sY\left(\frac{I}{K_p}\right)}{G(I, V)} - \delta \quad (\text{B17})$$

$$\frac{\dot{K}_h}{K_h} = \frac{sk^\alpha}{G[\theta(k), 1]} + \lambda + n = \frac{sY\left(\frac{V}{K_h}\right)}{G(I, V)} + \lambda + n \quad (\text{B18})$$

Repeating Equation (B14), after substituting Equations (B17)-(B18),

$$\frac{\dot{k}}{k} = \frac{sY\left(\frac{I}{K_p}\right)}{G(I, V)} - \frac{sY\left(\frac{V}{K_h}\right)}{G(I, V)} - (\lambda + n + \delta) \quad (\text{B19})$$

From Equations (B17)-(B19), when s is raised, for the $\frac{\dot{K}_p}{K_p}$ line to have an upward shift larger than the upward shift of the $\frac{\dot{K}_h}{K_h}$ line, or for the $\frac{\dot{k}}{k}$ line to shift upward, $\frac{I}{K_p}$ must be larger than $\frac{V}{K_h}$ for all k . Then the RHS of Equation (B19) is positive ($\frac{\dot{k}}{k} > 0$) when s increases, and for Equation (B12) to be zero (an implicit solution for k^*), k^* must increase since $\frac{d}{dk}\left(\frac{\dot{k}}{k}\right) = \frac{d}{dk}\left(\frac{\dot{K}_p}{K_p}\right) - \frac{d}{dk}\left(\frac{\dot{K}_h}{K_h}\right) < 0$ [Equation (B13)]. Therefore, if $\frac{I}{K_p} > \frac{V}{K_h}$ in the neighborhood of the steady state, then $\frac{\partial k^*}{\partial s} > 0$ as shown in Figure 2 in the main text.

The inequality $\frac{I}{K_p} > \frac{V}{K_h}$ can be roughly checked around the neighborhood of the steady state. This inequality can be restated as $\frac{[\frac{I}{Y}]}{[\frac{K_p}{Y}]} > \frac{[\frac{V}{Y}]}{[\frac{K_h}{Y}]}$, where $\frac{K_p}{Y} = k^{(1-\alpha)}$, $\frac{K_h}{Y} = k^\alpha$, so that $\frac{[\frac{K_p}{Y}]}{[\frac{K_h}{Y}]} = k^{(1-2\alpha)}$. Thus, the condition for $\frac{\partial k^*}{\partial s} > 0$ is that $\frac{I^*}{Y} > \left\{ [k^{*(1-2\alpha)}] \right.$ multiplied by $\left. \left(\frac{V^*}{Y} \right) \right\}$.

Box 1B reports the data to evaluate the above inequality. The data inputs are:

$$\frac{I^*}{Y} = 0.225; k^* = 2.5; \frac{V^*}{Y} = 0.077.$$

Two alternative values for α are used: 0.3 and 0.4. The results are:

For $\alpha=0.3$:

$$\frac{I^*}{Y} = 0.225 > 0.1334$$

For $\alpha=0.4$:

$$\frac{I^*}{Y} = 0.225 > 0.0924.$$

The inequality $\frac{\partial k^*}{\partial s} > 0$ is supported by the data in Box 1B.

Box 1B.

Data on k , $\frac{I}{Y}$, $\frac{V}{Y}$

This box reports the data on $k = \frac{K_p}{K_h}$ measured as capital services per hour for all private business during 1987-2018 (BLS, 2020), $\frac{I}{Y}$ = ratio of gross domestic capital formation to GDP (CEIC, 2020), $\frac{EDU}{Y}$ = ratio of total public and private expenditures on education to GDP (World Bank, 2020), $\frac{R\&D}{Y}$ = ratio of expenditures on research and development to GDP (OECD, 2020), $\frac{V}{Y} = \frac{EDU}{Y} + \frac{R\&D}{Y}$.

CEIC Notes on $\frac{I}{Y}$: United States Investment accounted for 20.4 % of its Nominal GDP in March 2020, compared with a ratio of 20.5 % in the previous quarter. United States investment share of Nominal GDP data is updated quarterly, available from March 1947 to March 2020, with an average ratio of 22.5 %. The data reached an all-time high of 25.4 % in December 1978 and a record low of 16.1 % in June 1947. CEIC calculates Investment as % of Nominal GDP from quarterly Nominal Gross Capital Formation and quarterly Nominal GDP. The Bureau of Economic Analysis provides Nominal Gross Capital Formation in USD and Nominal GDP in USD. Seasonally adjusted auxiliary series is used.

Description	1987- 2018	1987- 1990	1990- 1995	1995- 2000	2000- 2007	2007- 2018	2016- 2017	2017- 2018
k	2.5	2.1	2.1	3.7	3.4	1.7	1.3	0.7
$\frac{I}{Y}$ (average) (March 1947 – March 2020)	0.225							
$\frac{EDU}{Y}$ (2014)	0.050							
$\frac{R\&D}{Y}$ (2018)	0.027							
$\frac{V}{Y}$	0.077							