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MODELING HIGH DIMENSIONAL ASSET PRICING RETURNS USING A DYNAMIC SKEWED COPULA MODEL

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ABSTRACT

We propose a dynamic skewed copula to model multivariate dependence in asset returns in a flexible yet parsimonious way. We then apply the model to 50 exchange-traded funds. The new copula is shown to have better in-sample and out-of-sample performance than existing copulas. In particular, the dynamic model is able to capture increasing dependence patterns during financial crisis periods. It is crucial for investors to take dynamic dependence structure into account when modeling high dimensional returns.

Keywords: Skewed copula; Dynamic model; High dimensions; Multivariate dependence.
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I. INTRODUCTION

Modeling the dependence of a large number of financial asset returns is critical in financial applications such as asset allocation and risk management. In practice, the dependence of asset returns is shown to be time-varying. For example, in the 2008 subprime crisis, the financial markets were found to be more correlated and to have a contagion effect. But existing models often fail to capture the increased dependence and lead to an underestimation of portfolio risk during crisis periods. Hence, it is important for investors to model flexible dependence patterns of financial returns. In this context, copulas are widely used, as they provide more flexible dependence patterns than parametric distributions, such as multivariate normal or Student's t. As an example, investors can make use of copula models to evaluate the downside risk of a portfolio. Financial assets are more likely to crash together than boom together. This asymmetric dependence of asset returns is well captured by some copulas, and better understanding of asset return dependence improves estimates of portfolio risk. By contrast, linear correlation fails to capture this asymmetric feature and may lead to an underestimating downside risk in portfolio management. The literature provides numerous empirical studies on this topic (see, e.g., Patton (2006), Okimoto (2008), Wu and Liang (2011), and Elkamhi and Stefanova (2015)).

However, early applications of copulas are almost all bivariate in nature. That is, the copula is used to describe the dependence structure between two assets. It remains a challenge for academia and practitioners to model a highly flexible dependence structure among multivariate assets. When the number of assets is relatively large, models developed for low-dimension problems are often not applicable, either because the generalizations beyond the bivariate model are too restrictive, or because the simple generalization of the bivariate case leads to a proliferation of parameters and unmanageable computation complexity. For instance, Archimedean copulas are extremely restrictive in the multi-dimensional case, as they imply equicorrelated ranks. Another example is the N-dimensional Gaussian or t copula, whose parameters are difficult to estimate due to "curse of dimensionality" when N is large.[4] In high dimensional applications, we need to find a tradeoff between flexibility and parsimony.

Previous work on extending bivariate copula models to higher dimensions includes Genest, Gendron, and Bourdeau-Brien (2009) and Patton (2009), Christoffersen, Errunza, Jacobs, and Langlois (2012), and Gonzalez-Pedraz, Moren, and Pena (2015). We extend the model suggested in Christoffersen et al. (2012) and Gonzalez-Pedraz et al. (2015), and propose a dynamic skewed copula to model multi-variate dependence flexibly. Note that the so-called "dynamic" in our model refers to time variations in skewness parameters, while the skewness parameters in the aforementioned two copulas are static.

We make several contributions to the literature. First, the dynamic skewed copula can describe changing dependence patterns, since the skewness parameters follow an autoregressive procedure. By contrast, the skewness parameters in Christ

[4] High dimensional copulas in this paper refer to multivariate copula models describing the dependence of N time series when N is greater than 50. In Oh and Patton (2018), a low dimensional copula is defined as N no more than 5 and a high dimensional copula is defined as N between 50 and 250.

offersen et al. (2012) and Gonzalez-Pedraz et al. (2015) are constant, suggesting an unchanged dependence pattern throughout the sample period; that is, the upper tail dependence of all variables is always higher (or lower) than the lower tail dependence. Second, the dynamic skewed copula provides a parsimonious way to make each pair of variables display different dependence patterns. In addition, our model significantly reduces the number of parameters to be estimated while keeping flexibility in the model specification. Both Gonzalez-Pedraz et al. (2015) and our model specify N different skewness elements for N variables. However, the copula in Gonzalez-Pedraz et al. (2015) has N skewness parameters to be estimated and may encounter the "curse of dimensionality" problem, while our dynamic model avoids this problem with only five unknown parameters.

We apply our new model to investigate the dependence pattern of 50 U.S. exchange-traded fund (ETF) returns. Our dynamic skewed copula provides better goodness-of-fit than those in Christoffersen et al. (2012) and Gonzalez-Pedraz et al. (2015). In particular, the dynamic settings of skewness parameters enable us to characterize different dependence patterns in crisis periods and in tranquil periods. This implies that financial asset returns tend to exhibit time-varying rather than static dependence patterns. Thus, for investors, it is inappropriate to stick to one dependence structure when modeling high dimensional financial returns.

This paper proceeds as follows. Section 2 introduces the dynamic skewed copula model and explains how to estimate it. Section 3 provides summary statistics of ETF returns. Section 4 analyzes the in-sample dependence structure of these ETF returns. Out-of-sample performance of the model is presented in Section 5. Finally, Section 6 sets forth our conclusions.

II. THE DYNAMIC SKEWED COPULA MODEL

The dynamic skewed copula model discussed in this paper originates from the skewed t copula developed by Christoffersen et al. (2012). Let $(u_{1t}, \cdots, u_{Nt})$ denote the cumulative distribution functions (CDF) of N variables, t = 1, $\cdots$, T . Their dependence is described by a copula function $C(u_{1t}, \cdots, u_{Nt})$. The skewed t copula in Christoffersen et al. (2012), denoted by C^{skt} is given by

$$C^{skt}(u_{1t}, \dots, u_{Nt}; R_t, v, \bar{\gamma}) = H_{R_t,v,\bar{\gamma}}(H_{v,\bar{\gamma}}{}^{-1}(u_{1t}), \dots, H_{v,\bar{\gamma}}{}^{-1}(u_{Nt})) \tag{1}$$

where R_t is the correlation matrix, v is the degrees of freedom, $\bar{\gamma}$ is the skewness parameter, which is a scalar. $H_{R_t,v,\bar{\gamma}}(\cdot)$ is the CDF of N-dimensional generalized hyperbolic skewed t distribution with zero mean as in Demarta and McNeil (2005). For $i = 1, \cdots, N$, $H_{v,\bar{\gamma}}{}^{-1}(\cdot)$ is the inverse of univariate skewed t CDF on the i^{th} subordinate with zero mean and unit variance. Please refer to Christoffersen et al. (2012) for details of $H_{v,\bar{\gamma}}{}^{-1}(\cdot)$.

The evolution of R_t is similar to the dynamic mechanism in the Engle (2002) dynamic conditional correlation (DCC) model.

$$R_t = diag\{Q_t\}^{-1/2} Q_t \, diag\,\{Q_t\}^{-1/2} \tag{2}$$

$$Q_t = (1-\alpha_1 - \alpha_2)\bar{Q} + \alpha_1 z_{t-1} z'_{t-1} + \alpha_2 Q_{t-1},$$

where $\overline{Q}$ is the unconditional covariance matrix of $z_t = (z_{1t},..., z_{Nt})'^1$. For $i = 1,\cdots, N$, $z_{it} = H_{v,\overline{\gamma}}^{-1}(u_{it})$. $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 < 1$, the conditional correlation R_t is mean-reverting. This model C^{skt} has four parameters $\theta^{skt} = (\alpha_1, \alpha_2, v, \overline{\gamma})'$.

Note that in Christoffersen et al. (2012), the skewness parameter $\overline{\gamma}$ is a scalar rather than a vector, indicating that all variables have identical asymmetry parameters. There are two problems with this specification. First, it is unable to describe non-exchange dependence patterns such that $C(\cdots, u_{it}, \cdots, u_{jt}, \cdots) \neq C(\cdots, u_{jt}, \cdots, u_{it}, \cdots)$ for $\forall i \neq j$. Second, it is unable to capture the time variation in dependence due to the constant skewness parameter. In reality, the multivariate dependence of financial assets may be time-varying over time.[5]

To overcome these problems, we modify the copula in Christoffersen et al. (2012) via the following two steps. First, we assume each marginal CDF u_{it} corresponds to a different skewness parameter. The skewness parameter now becomes an N-dimensional vector rather than a scalar. In this case, the multivariate skewed t copula, denoted by C^{mskt}, is written as:

$$C^{mskt}(u_{1t}, \dots, u_{Nt}; R_t, v, \overline{\gamma}) = H_{R_t,v,\gamma}(H_{v,\gamma_1}{}^{-1}(u_{1t}), \dots, H_{v,R_N}{}^{-1}(u_{Nt}) \quad (3)$$

The skewness parameter $\gamma = (\gamma_1 \dots \gamma_N)'$ is an N-dimensional vector, but is still time-invariant. The definitions of R_t and v are the same as those for C^{skt}. The parameters in C^{mskt} are $\theta^{skt} = (\alpha_1, \alpha_2, v, \gamma')'$, so $N + 3$ parameters are to be estimated. We are not the first to propose C^{mskt}, as Gonzalez-Pedraz et al. (2015) use this model to study the dependence of three assets: oil futures, gold futures, and the S&P 500 equity index. Clearly, simply extending C^{skt} to C^{mskt} is applicable only for small N, such as $N = 3$ in Gonzalez-Pedraz et al. (2015). For large N, it is difficult to estimate all the parameters, due to the "curse of dimensionality." In practice, to reduce the number of unknown parameters, we can group the $\gamma_i s$ and set the $\gamma_i s$ within the same group to be equal. But such a specification is very subjective and lacks statistical justification. We will use this type of constrained C^{mskt} in our empirical analysis of ETFs.

The second step is to specify a dynamic evolution process for the skewness vector. The dynamic mechanism is similar to the Engle (2002) DCC model. The modified model, called dynamic skewed copula C^{dskt}, is given by

$$C^{dskt}(u_{1t}, \dots, u_{Nt}; R_t, v, \overline{\gamma}) = H_{R_t,v,\overline{\gamma}}(H_{v,\overline{\gamma}}{}^{-1}(u_{1t}), \dots, H_{v,\overline{\gamma}}{}^{-1}(u_{Nt}), \quad (4)$$

$$\gamma_t = (1 - \beta_1 \frac{v}{v-2} - \beta_2)\overline{\gamma} + \beta_1 z_{t-1} + \beta_2 \gamma_{t-1} \quad (5)$$

where β_1 is the coefficient of data-driven term z_{t-1}, and β_2 lies within (-1, 1) to ensure γ_t is mean-reverting. Since $E(z_t) = E(z_{t-1}) = \frac{v}{v-2}\overline{\gamma}$, the coefficient in front of $\overline{\gamma}$ is $1-\beta_1 \frac{v}{v-2} - \beta_2$. Note that both the copula correlation matrix R_t and skewness vector γ_t are defined based on the copula shocks $z_{it} = H_{v,\overline{\gamma}}{}^{-1}(u_{it})$, rather than the return shocks (standardized residuals ϵ_{it} from marginal distributions).

[5] In Christoffersen et al. (2012), Qt is assumed to be mean-reverting at $(1-\phi)Q + \phi\Gamma t$, a weighted average of Q and a time-varying matrix Γt containing time-trend information. Here, we ignore the time trends in Qt by setting the weighting parameter $\phi = 0$.

The dynamic skewed copula C^{dskt} has five parameters $\theta^{dskt} = (\alpha_1, \alpha_2, v, \beta_1, \beta_2)'$. By introducing dynamics into the skewness vector, C^{dskt} has great flexibility in capturing multivariate dependence with only a few parameters. The model has two desirable features. First, it allows each pair of assets to display dependence patterns that are distinct from other pairs. Second, it is able to describe changing dependence patterns over the time.

For the estimation procedure, we mainly follow Christoffersen et al. (2012). The only difference is that the copula's skewness parameter is an N-dimensional vector in C^{mskt} and is an autoregressive vector in C^{dskt}. The joint distributions of financial returns are estimated via the following two steps.

First, we estimate each univariate marginal model and calculate the marginal CDF u_{it} for $i = 1, \cdots, N$, $t = 1, \cdots, T$. Second, we estimate copula models by maximum composite likelihood estimation (MCLE). MCLE is employed because it yields consistent estimates for the true parameters in large scale problems, while the ordinary maximum likelihood estimation (MLE) may estimate the parameters driving the dynamic process with bias, as discussed in Engle, Shephard, and Shepphard (2008), Christoffersen et al. (2012), and Oh and Patton (2015).

For $k = skt, mskt, dskt$, the composite log-likelihood of copula $C^k(u_{1t}, \cdots, u_{Nt}; \theta^k)$ is defined as

$$CL(\theta^k) = \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{i>j} ln c^k(u_{it}, u_{jt}; \theta^k), \tag{6}$$

where $C^k(u_{it}, u_{jt}; \theta^k)$ denotes the bivariate copula density of pair i and j for $i, j=1, \cdots, N$.

III. DATA DESCRIPTION

Our empirical analysis employs the dynamic skewed copula model to study the dependence of 50 US ETF returns ($N = 50$). The data set includes daily adjusted prices of four stock ETFs (STK) and five other types of ETF: bond (AGG), foreign exchange (Euro/Dollar, FXE), gold (GSG), oil (USO), and real estate (RWR).[6] Stocks are selected from nine sectors, and in each sector only the top five firms with the highest market values are considered. All prices are in US dollars and are from Bloomberg. The i^{th} ($i= 1, \cdots, N$) daily return is calculated as $r_{it} = 100 \times (logP_{it} - logP_{it-1})$, where P_{it} is the closing price of ETF i on day t.

The sample period is from July 24, 2006 to June 28, 2013, for a total of 1746 daily observations. We divide the sample into two subperiods, such that the period from July 24, 2006 to June 30, 2011 is used for the in-sample estimation (1245 observations), and the remaining 501 observations, from July 1, 2011 to June 28, 2013, are reserved for the out-of-sample portfolio performance evaluation.

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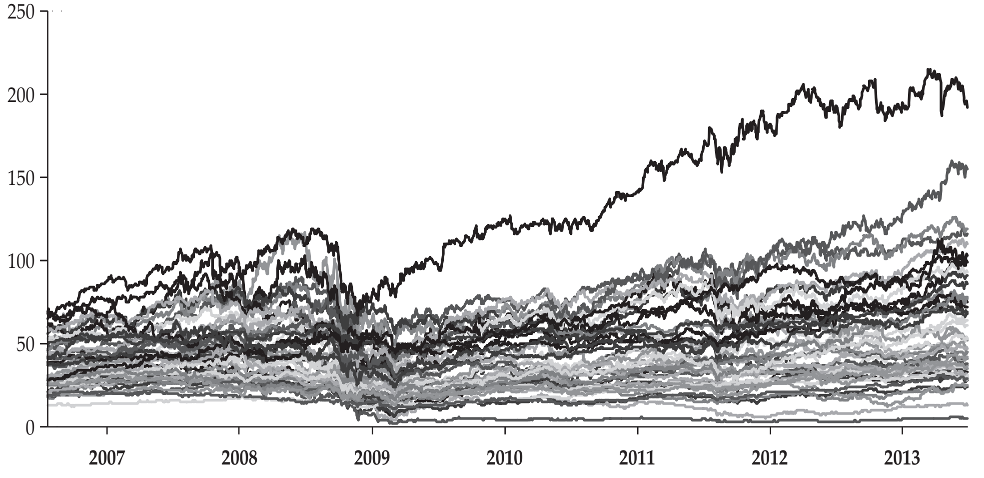
[6] Daily prices are adjusted for dividends. Calculation of dividends is as follows. For example, Materials Select Sector SPDR Fund (XLB) distributes $0.239 dividend per share on March 14, 2019. The closing price of XLB is $55.57, but the adjusted closing price is $55.239. We then use adjusted prices to account for the impacts of dividend distribution on stock prices.

Figure 1. Price Series of the 50 ETFs

Panel (a) plots 45 stock ETFs prices over the sample period of July 24, 2006 - June 28, 2013. Panel (b) plots the ETF prices of bond (AGG), foreign exchange ETF (FXE), gold (GSG), oil (USO) and real estate (RWR) over the same sample period.

(a) 45 stock ETFs



(b) 5 ETFs: bond, foreign exchange, gold, oil and real estate

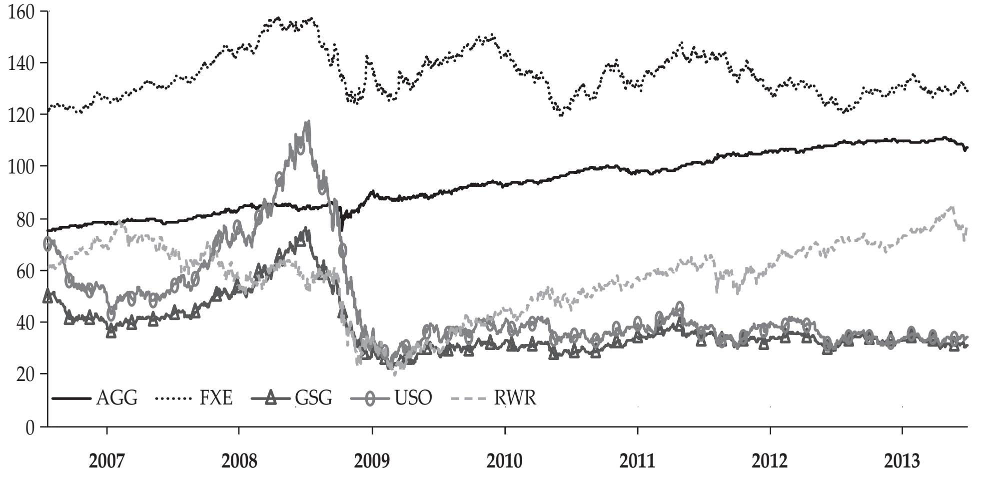


Figure 1 plots the price series of the 50 ETFs. The movements of all stock ETFs and the real estate ETF are quite similar, while the behavior of the bond ETF generally differs from the others. These ETFs were greatly impacted by the 2008 subprime crisis, as nearly all of them (except the bond ETF) suffered from a 9-month decrease in prices following the crisis.

Table 1.
Descriptive Statistics and Tests for Time-Varying Dependence

Panel A summaries the descriptive statistics of 14 representative ETF returns over the sample period of July 24, 2006 - June 30, 2011. Std is the standard deviation. Kurtosis is the excess kurtosis. J-B is the Jarque-Bera test for normality. Ljung-Box(5) and ARCH(5) are the Ljung-Box tests for serial correlation and for ARCH effects with 5 lags. The ***, ** and * denote the significance level at 1%, 5%, and 10%, respectively. Panel B gives the proportion of ETF pairs that have time-varying rank correlations within 45 stock ETFs (column 1), between 45 stocks and 5 other ETFs (column 2), within 5 other ETFs (column 3), and within all 50 ETFs (column 4). Based on Patton (2012), two ETFs, i and j, are expected to have time-varying dependence if the autoregressive vector $(\rho_1, \rho_2, \ldots \rho_p)'$ is jointly significant in the following model: $u_{it}\, u_{jt} = \rho_0 + \Sigma^{p}_{p=1}\, \rho_p\, U_{it\text{-}P}\, U_{jt\text{-}P}$ with lags p = 1 or p = 5.

	Panel A: In-sample descriptive statistics						
	Mean	**Std.**	**Skewness**	**Kurtosis**	**J-B (10^3)**	**Ljung-Box (5)**	**ARCH (5)**
Stocks							
XLB(DD)	0.0118	2.8076	-0.3779	6.5718	2.2558***	44.4638***	85.3496***
XLE(XOM)	0.0254	2.3042	-0.4035	6.9109	2.4959***	85.3132***	43.2086***
XLF(WFC)	-0.0086	3.9549	0.7616	12.2928	7.9151***	92.0241***	63.6810***
XLI(GE)	0.004	2.1224	0.1956	4.1057	0.8761***	46.6722***	56.1684***
XLK(MSFT)	0.0166	1.9895	0.1033	6.135	1.9422***	52.9964***	45.6344***
XLP(WMT)	0.0464	1.36	0.7098	12.8008	8.5571***	63.2657	43.5734***
XLU(DUK)	0.0351	1.2326	0.7873	9.3449	4.6317***	126.4957***	38.0077**
XLV(JNJ)	0.0203	1.3694	-0.1759	6.0569	1.8973***	55.7293***	53.7298***
XLY(DIS)	0.0176	2.1042	0.4836	3.9309	0.8441***	42.3541**	82.2823
Bond (AGG)	0.0235	0.4209	-3.1702	73.4637	280.6589***	140.5175***	35.9942*
Exchange (FXE)	0.0135	0.7072	0.033	2.177	0.2438***	52.3304***	60.0650***
Gold (GSG)	-0.031	1.8638	-0.3957	2.171	0.2746***	29.937	60.0417***
Oil (USO)	-0.0522	2.4324	-0.2126	2.0082	0.2165***	34.5330*	70.1851***
Real estate (RWR)	0.0027	3.0878	-0.2206	7.991	3.30278***	131.3786***	85.6806***

	Panel B: Tests for time-varying pairwise dependence			
	45 stocks	45 stocks-5 others	5 others	50 ETFs
p = 1	0.9535(= 944/990)	0.8800(= 220/225)	1.0000(= 10/10)	0.9584(= 1174/1225)
p = 5	1.0000(= 990/990)	1.0000(= 225/225)	1.0000(= 10/10)	1.0000(= 1225/1225)

Table 1 provides preliminary analyses of the in-sample data. Panel (a) reports summary statistics for 14 representative ETFs: 9 stock ETFs, each of which has the highest market value in its own sector, and 5 ETFs of other asset types, including bonds, foreign exchange, gold, oil, and real estate. Most ETFs yield positive returns, while commodity ETFs (gold and oil) have negative returns. Stock, commodities, and real estate ETFs tend to be more volatile than bond and foreign exchange ETFs. Further, most returns exhibit non-normal features, as in Alexander and Barbosa (2008) and Hsu, Hsu and Kuan (2010). Panel (b) gives the proportion of ETF pairs that display time-varying rank correlations. In most cases, we find evidence of dynamically evolving measures of dependence among the ETFs. These findings support application of our dynamic skewed t copula to describe changing dependence patterns in the following analysis.

IV. In-Sample Analysis

We first estimate the marginal distribution of each ETF return series. Then, we investigate the contemporaneous dependence of these returns based on the dynamic skewed copula. Finally, we select four representative asset pairs to further illustrate the flexibility of C^{dskt} in modeling multivariate dependence.[7]

Table 2.
Marginal Model Estimates Over The In-Sample Period

This table reports the marginal model estimates of 14 representative ETF returns over the sample period of 2006-06-24 to 2011-06-30. Standard errors are given in the brackets below the parameters. The ***, **and* denote the significance at 1%, 5%, and 10% levels, respectively. Log L denotes the log-likelihood value of each marginal model. K-S is the Kolmogorov-Smirnov test with the null hypothesis that the model is correctly specified.

Stocks	$\mu_i(10^{-3})$	ρ_i	$_{ki0}(10^{-3})$	k_{i1}	k_{i2}	k_{i3}	δi	f_{i1}	f_{i2}	logL	K-S
XLB(DD)	-0.1265	-0.0699*	0.0219***	0.0706***	0.9294***	0.6944***	0.5099***	2.5803***	3.6634***	4.5237	0.0211
	(0.2768)	(0.0431)	(0.0080)	(0.0094)	(0.0117)	(0.1086)	(0.0115)	(0.3005)	(0.4754)		
XLE(XOM)	-0.2435	0.0935*	0.1229***	0.1145***	0.8855***		0.5170***	3.3588***	8.2809***	4.9015	0.0241
	(0.3381)	(0.0431)	(0.0080)	(0.0094)	(0.0117)		(0.0115)	(0.3005)	(0.4754)		
XLF(WFC)	0.0741	-0.1167**	0.0067*	0.1128***	0.8872***	0.8015***	0.4931***	2.9330***	2.5219***	4.1069	0.0203
	(0.0495)	(0.0504)	(0.0037)	(0.0116)	(0.0335)	(0.1829)	(0.0115)	(0.3975)	(0.3368)		
XLI(GE)	0.0009	0.0046	0.0802***	0.0843***	0.9157***		0.5046***	3.4614***	4.2664***	4.9888	0.0205
	(0.0082)	(0.0301)	(0.0116)	(0.0136)	(0.0110)		(0.0109)	(0.4260)	(0.6896)		
XLK(MSFT)	-0.1729	-0.0392	0.0869**	0.1188***	0.8812***		0.5127***	2.9879***	4.5885***	5.1598	0.0228
	(0.2695)	(0.0348)	(0.0344)	(0.0390)	(0.0391)		(0.0128)	(0.6102)	(2.2651)		
XLP(WMT)	-0.5015	-0.0798	0.0336***	0.1161***	0.8839***	0.0017*	0.4976***	3.5831***	3.4657***	5.9806	0.0187
	(0.3937)	(0.0631)	(0.0058)	(0.0131)	(0.0159)	(0.0009)	(0.0111)	(0.4955)	(0.4444)		
XLU(DUK)	0.4004	-0.1648***	0.0513***	0.1738***	0.8257***	0.0054***	0.4975***	4.1372***	4.1854***	6.1542	0.019
	(0.2860)	(0.0551)	(0.0080)	(0.0189)	(0.0223)	(0.0009)	(0.0112)	(0.6493)	(0.6022)		
XLV(JNJ)	-0.1999	-0.035	0.0421	0.0911***	0.9086***	0.0088**	0.5007***	4.6611***	4.5403***	5.8591	0.0202
	(0.2333)	(0.0363)	(0.0079)	(0.0124)	(0.0146)	(0.0040)	(0.0104)	(0.8491)	(0.7755)		
XLY(DIS)	0.1571	0.0088	0.1380***	0.1673***	0.8327***		0.4893***	4.4986***	3.0871***	5.0272	0.0207
	(0.4224)	(0.0153)	(0.0183)	(0.0150)	(0.0169)		(0.0117)	(0.7652)	(0.3943)		
Bond (AGG)	-0.2692**	-0.1366	0.0005***	0.0673***	0.9324***	0.7365***	0.5145***	3.6681***	12.1769***	8.4846	0.0239
	(0.1354)	(0.1100)	(0.0001)	(0.0135)	(0.0184)	(0.1756)	(0.0101)	(0.6527)	(4.6478)		
Exchange	-0.1365	0.0482	0.0003**	0.0259***	0.9739***	0.8808***	0.5009***	6.2868***	6.5946***	7.1463	0.0207
(FXE)	(0.1504)	(0.0356)	(0.0002)	(0.0088)	(0.0117)	(0.1890)	(0.0103)	(1.4981)	(1.6376)		
Gold (GSG)	0.3073	-0.0342	0.1148***	0.1064***	0.8932***	0.0039**	0.5124***	4.5210***	11.9471***	5.2025	0.0232
	(1.3851)	(0.0342)	(0.0151)	(0.0123)	(0.0123)	(0.0018)	(0.0103)	(0.7678)	(4.1833)		
Oil (USO)	0.5186	-0.0411	0.1832***	0.0985***	0.9015***	0.0023***	0.5081***	5.8351***	10.5864***	4.6771	0.0217
	(0.6897)	(0.0359)	(0.0256)	(0.0106)	(0.0118)	(0.0005)	(0.0103)	(1.2147)	(3.3274)		
Real estate	-0.0381**	-0.2210***	0.0154***	0.0947***	0.9053***	0.7819***	0.5078***	3.2109***	4.1998***	4.4879	0.0214
(RWR)	(0.0152)	(0.0509)	(0.0041)	(0.0079)	(0.0160)	(0.0952)	(0.0111)	(0.4349)	(0.7066)		

[7] Details of these 50 ETFs are available upon request.

In Table 2, we estimate the marginal distributions of ETF returns via the autoregressive and nonlinear GARCH models (AR-NGARCH) with the generalized asymmetric Student's t (AST) errors from Zhu and Galbraith (2011). Such specifications are able to capture the non-normal features illustrated in Table 1. For simplicity, only 14 marginal models are reported.

The conditional mean and conditional volatility models are as follows:

$$r_{it}=\mu_i+\rho_i r_{it-1}+\sqrt{h_{it}}\epsilon_{it}, \quad \epsilon_{it}|\ \Omega_{it-1}\sim i.i.d.\,AST(\delta_i) \tag{7}$$

$$h_{it} = k_{i0} + k_{i1}h_{it-1}(\epsilon_{t-1} - k_{i3})^2 + k_{i2}h_{it-1} \tag{8}$$

For AST distribution, δ_i is the skewness parameter, f_{i1} and f_{i2} represent the degrees of freedom on, respectively, the left side and right side of stochastic error ϵ_{it}.

Two conclusions can be drawn from Table 2. First, the results of the Kolmogorov–Smirnov test indicate that the marginal models are correctly specified. This ensures the consistency of copula estimates in the following subsection. Second, the parameter estimates in marginal models confirm the non-normal features of return series, including serial correlation, volatility clustering, leverage effects, and differing levels of thicknesses in the lower and upper tails. Among these features, positive k_{i3} indicates the presence of a leverage effect in most ETFs, but in Gonzalez-Pedraz et al. (2015), a leverage effect is found only in stocks, but not in oil. Besides, f_{i1} is generally lower than f_{i2}, illustrating a higher probability of crashing than booming for each ETF return.

We next transform the standardized residuals $(\hat{\epsilon}_{1t}, \cdots, \hat{\epsilon}_{Nt})$ into $(\hat{u}_{1t}, \cdots, \hat{u}_{Nt})$ and estimate the copula models. The results of five dynamic copulas are reported in our analysis: $C^{Gaussian}$, C^{t}, C^{skt}, C^{mskt}, and C^{dskt}. The last three copulas are given in equations (1), (3), and (4). To make the analysis more complete, we also provide the results of Gaussian and *t* copulas, as they may be regarded as special cases of skewed *t* copulas. Note that C^{mskt} here differs from the copula in Gonzalez-Pedraz et al. (2015). For C^{mskt}, we assume the skewness parameters of 4 stock ETFs are identical, since the behavior of these stock ETFs are quite similar. This specification avoids the "curse of dimensionality" problem for large *N* and simplifies the calculations.

The in-sample comparison between various copula models is based on the following criteria:

$$logL={}^{1}/_{T}\ \left[\sum_{t=1}^{T}\sum_{i=1}^{N} log f_i(r_{it}) + \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{i>j} log\, c(u_{it},u_{jt})\right],$$

$$\text{AIC} = -2log\,{}^{L}/_{T} + 2\text{K/T}$$

$$\text{SIC} = -2log\,{}^{L}/_{T} + \text{Kln(K)/T}$$

where 1og f_i is the log-likelihood of each marginal model, logc is the log-likelihood of bivariate copula for the i^{th} and j^{th} ETFs, *K* is the dimension of parameters, and *T* is sample size.

Table 3.
Copula Model Estimates Over the In-Sample Period

This table reports the estimates of copula models over the sample period of 2006-06-24 to 2011-06-30. Standard errors are given in the brackets below the parameters. The ***, **, and * denote the significance at the 1%, 5%, and 10% levels, respectively. Log L denotes the log-likelihood value of each model, including both copula and marginal models.

	Gaussian	t		skt		mskt		dskt
$\alpha 1$	0.0070***	0.0040***		0.0046***		0.0169***		0.0699***
	(0.0009)	(0.0005)		(0.0007)		(0.0014)		(0.0248)
$\alpha 2$	0.9504***	0.9480***		0.9367***		0.9614***		0.6781**
	(0.0080)	(0.0106)		(0.0162)		(0.0041)		(0.2831)
v		12.2706***		21.5303***		16.2605***		11.1504***
		(0.5155)		(1.3128)		(1.4157)		(2.3429)
			$\bar{\gamma}$	0.4259***	γ_{STK}	0.3588***	**$\beta 1$**	0.7326***
				(0.0137)		(0.0256)		(0.0309)
					γ_{AGG}	0.1934***	**$\beta 2$**	0.7822***
						(0.0616)		(0.0021)
					γ_{FXE}	0.0933*		
						(0.0544)		
					γ_{GSG}	0.2142***		
						(0.0605)		
					γ_{USO}	0.1924***		
						(0.0617)		
					γ_{RWR}	0.3574***		
						(0.0413)		
logL								
($\times 10^2$)	-1.7951	-1.7825		-1.7806		-1.7729		-1.7635
AIC								
($\times 10^2$)	3.5903	3.5651		3.5614		3.5459		3.5271
SIC								
($\times 10^2$)	3.5908	3.5655		3.5619		3.5463		3.5276

Table 3 presents in-sample estimates of the copula models. The dynamic skewed t copula C^{dskt} provides the best in-sample goodness-of-fit among the five models, as it has the lowest AIC and SIC. The values of AIC and SIC decrease when we switch from C^{skt} to C^{mskt}, implying the necessity to consider multiple skewness parameters. From C^{skt}, all 50 ETFs correspond to an identical skewness parameter $\bar{\gamma} = 0.43$. This implicitly assumes that these ETFs are more likely to boom together than crash together. This is, however, inconsistent with the findings in existing studies. In Patton (2004), Hong, Tu and Zhou (2007), and Rodriguez (2007), among others, financial assets are more correlated in market downturns than in upturns. Further, the values of AIC and SIC drop further if we switch from C^{mskt} to C^{dskt}, suggesting that a time-varying dependence pattern is more appropriate for these assets. In other words, the dependence structure of these 50 ETFs can change over time. In some periods, these ETF returns are more correlated when they increase together than when they decrease together, while in other periods, such as crisis periods, they are more correlated when they decrease together than when they increase together. The changing dependence patterns are discussed in more details below.

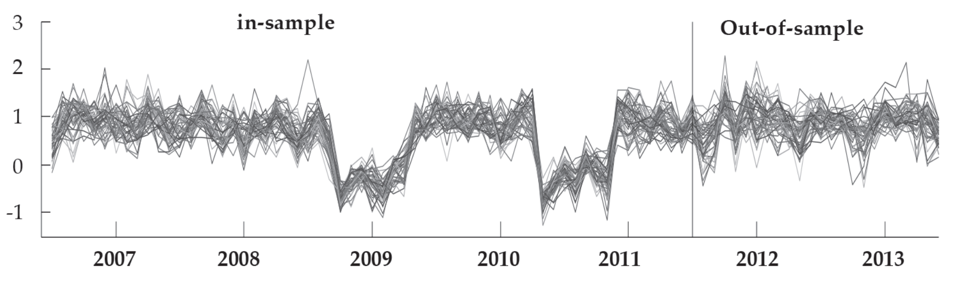
Estimates of C^{dskt} show that skewness vector γ_t is not only autoregressive (β_2= 0.78), but also negatively correlated with shocks from dependence system at t -1 (β_{12}= -0.73).

Comparing C^t and C^{dskt}, we observe that the autoregressive coefficients in conditional correlation α_2 and degree of freedom v are smaller if the dynamics of skewness parameter are considered. The decrease in α_X implies that the dynamic dependence structure is captured not only by the evolution of conditional correlation, but also by the evolution of the skewness parameter. The decrease in v indicates stronger tail dependence in C^{dskt} than in C^{skt}. Hence, C^{dskt} with a dynamic skewness vector is more likely to capture the dependence structure under extreme market conditions.

Figure 2. Skewness Parameter Estimates Series of 50 ETFs Based on C^{dskt}

The figures plot the skewness vector γ_t in equation (5) based on the dynamic skewed copula model C^{dskt}. Panel (a) is for the 45 stock ETFs, and panel (b) to (f) are for the ETFs of bond (AGG), foreign exchange ETF (FXE), gold (GSG), oil (USO) and real estate (RWR). The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.

(a) 45 stocks



(b) Bond

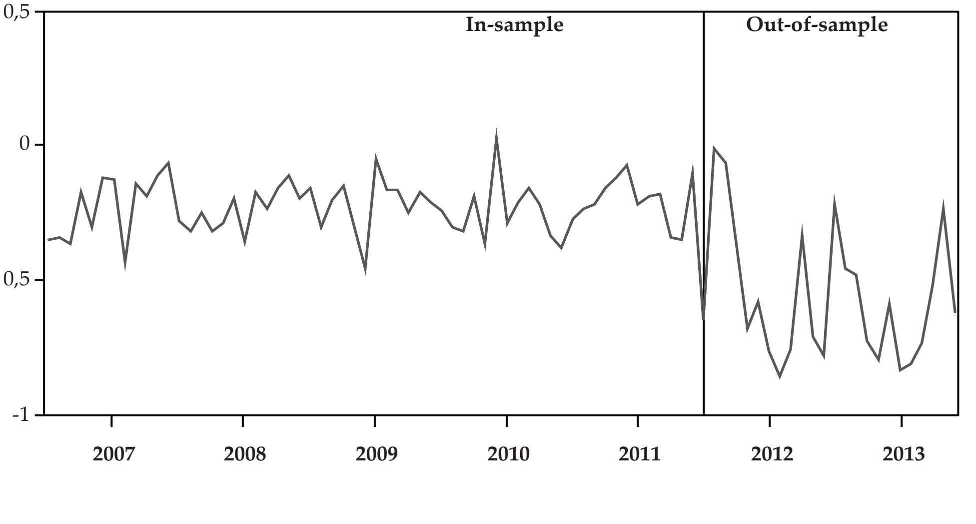
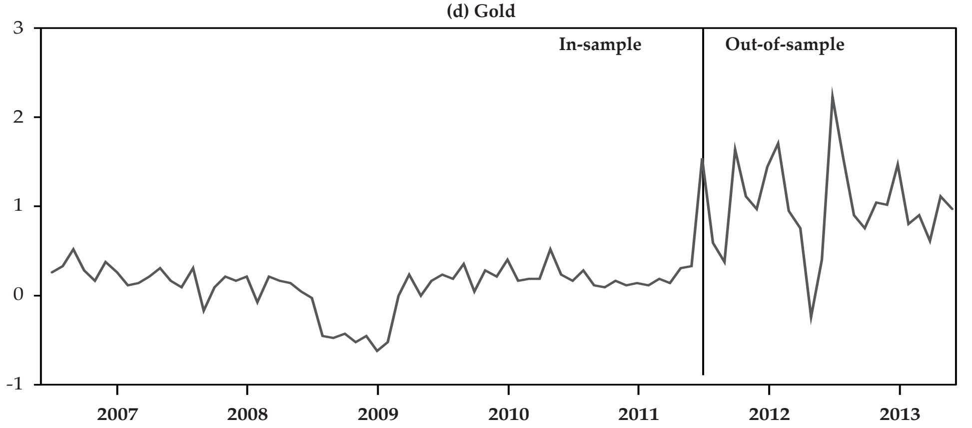


Figure 2. Skewness Parameter Estimates Series of 50 ETFs Based on C^{dskt} (contd.)

The figures plot the skewness vector γ_t in equation (5) based on the dynamic skewed copula model C^{dskt}. Panel (a) is for the 45 stock ETFs, and panel (b) to (f) are for the ETFs of bond (AGG), foreign exchange ETF (FXE), gold (GSG), oil (USO) and real estate (RWR). The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.

(c) Foreign exchange

(d) Gold



(e) Oil

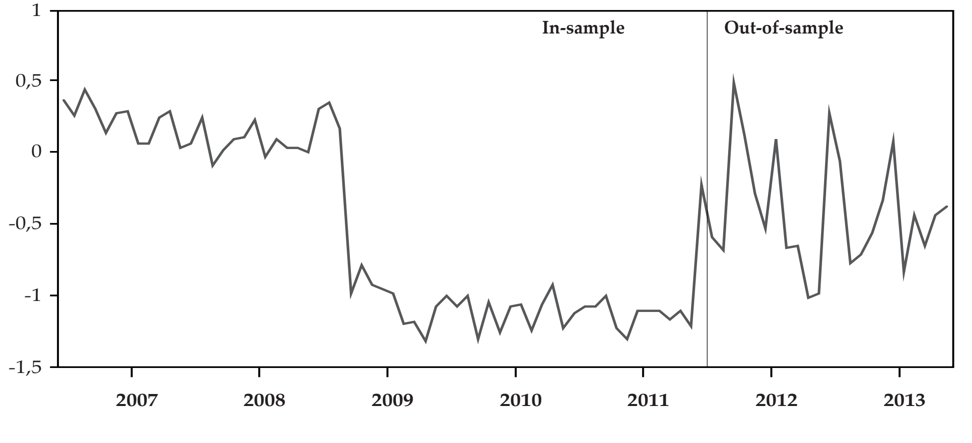


Figure 2. Skewness Parameter Estimates Series of 50 ETFs Based on C^{dskt} (contd.)

The figures plot the skewness vector γ_t in equation (5) based on the dynamic skewed copula model C^{dskt}. Panel (a) is for the 45 stock ETFs, and panel (b) to (f) are for the ETFs of bond (AGG), foreign exchange ETF (FXE), gold (GSG), oil (USO) and real estate (RWR). The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.

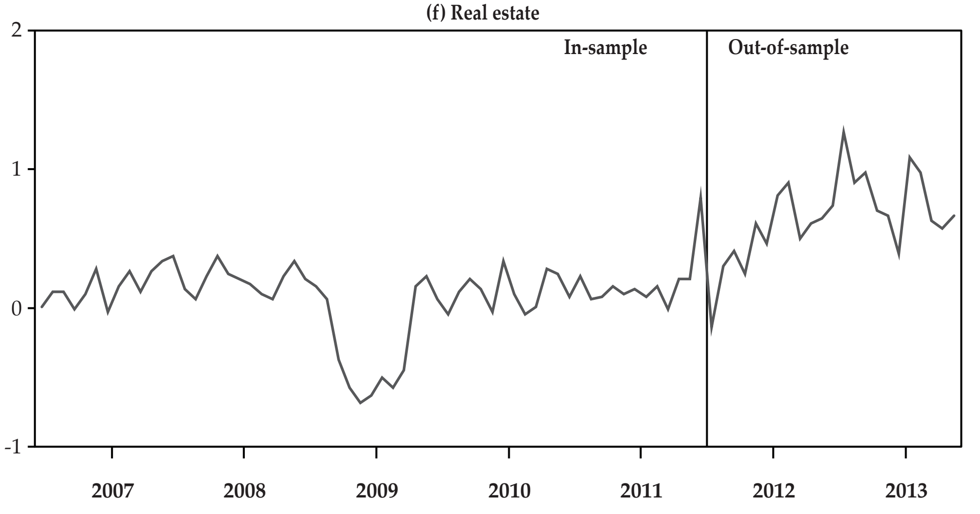
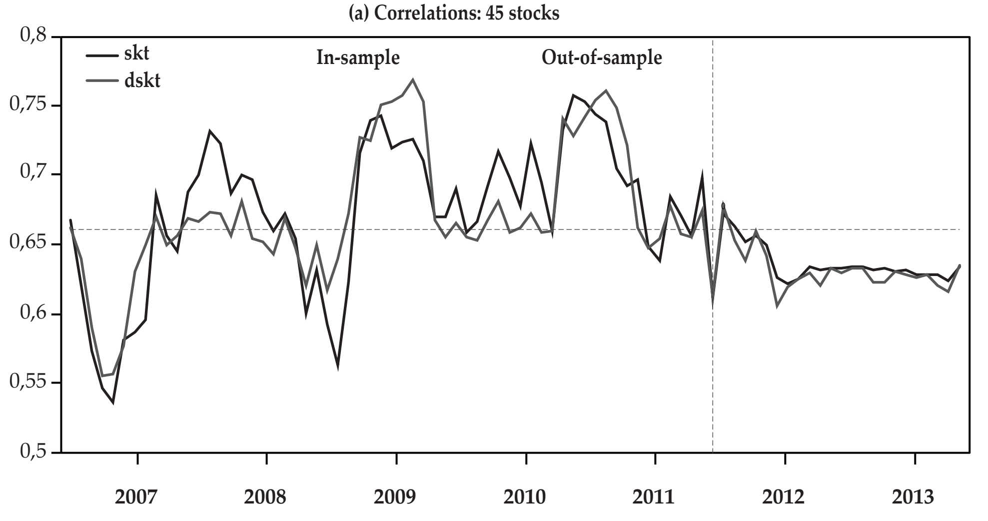


Figure 2 plots the skewness parameter series of 50 ETFs based on C^{dskt}. In Figure 2 (a), the 45 stock ETFs have similar skewness parameters, which are positive most of the time and become negative during the 2008 subprime crisis and the 2010 European debt crisis. In Figure 2 (b), the bond ETF generally has negative skewness parameters, whose behavior differs from other ETFs. In Figure 2 (c), (d), and (f), the skewness parameters of foreign exchange, gold, and real estate ETFs are positive in most cases, but decrease dramatically to become negative during the 2008 subprime crisis. In Figure 2 (e), the skewness parameters of oil ETFs exhibit a two-stage feature: positive before the 2008 subprime crisis and negative afterwards. Overall, Figure 2 shows the availability of our dynamic skewed *t* copula C^{dskt} in describing the dynamics of multivariate dependence patterns; it is more much flexible than the C^{skt} model in Christoffersen et al. (2012) with only a single skewness scalar.

Since it would be tedious to analyze all pairwise dependences among the 50 ETFs, we select four representative groups for further discussion: (1) 45 stocks; (2) stocks and bonds; (3) stocks and oil; and (4) oil and gold.

Figure 3. Average Correlation for 45 Stocks, Bond, Oil, Exchange Rate, and Gold

The figures plot the correlations R_t based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt}. Panel (a) is the average bivariate correlations across 990 pairs of stocks, panel (b) is the average correlations between each stock and bond across 45 pairs, panel (c) is the average correlations between each stock and oil across 45 pairs, and panel (d) is the correlations of foreign exchange and gold. The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.



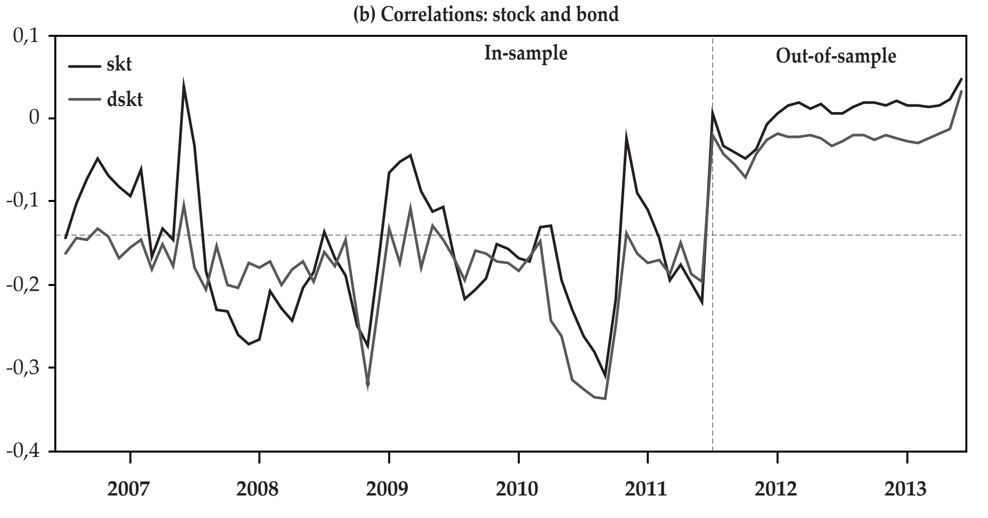
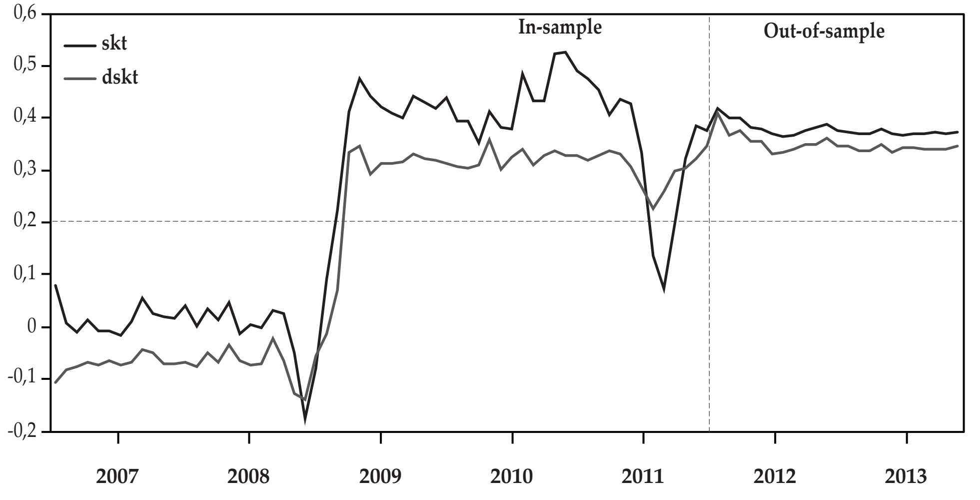


Figure 3. Average Correlation for 45 Stocks, Bond, Oil, Exchange Rate, and Gold (contd.)

The figures plot the correlations R_t based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt}. Panel (a) is the average bivariate correlations across 990 pairs of stocks, panel (b) is the average correlations between each stock and bond across 45 pairs, panel (c) is the average correlations between each stock and oil across 45 pairs, and panel (d) is the correlations of foreign exchange and gold. The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.

(c) Correlations: stock and oil



(d) Correlations: foreign exchange and gold

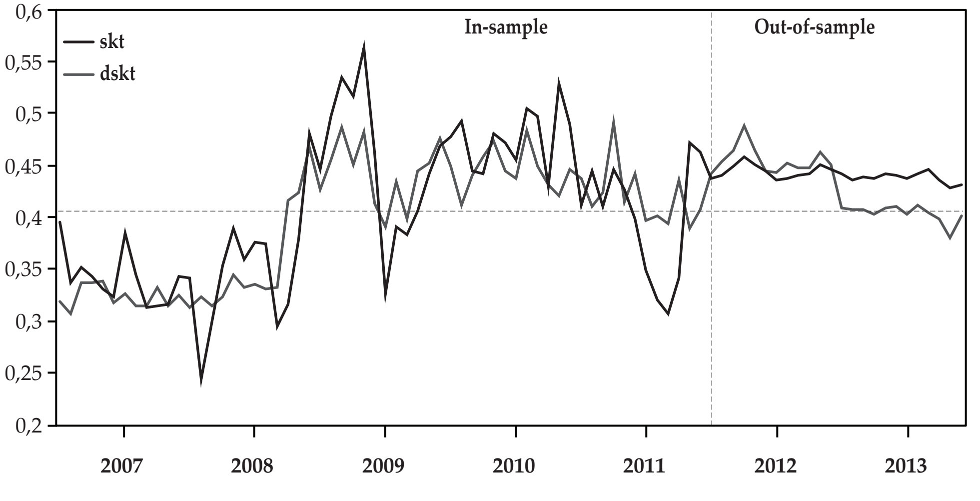


Figure 3 and Figure 4 plot the average correlations and exceeding correlations (at 10% and 90% quantiles) within each group. Both the results of the Christoffersen et al. (2012) C^{skt} and our C^{dskt} are provided to better understand the dynamics in dependence patterns. The in-sample portion is estimated from the in-sample data, and the out-of-sample portion is calculated via one-step-ahead forecast with rolling windows (see Section 5). Note that, for group (2), the exceeding correlation

$\tilde{\rho}^*_{STK,AGG,t}$ (0.1) measures the correlation when stock return decreases below its 10% quantile and bond return increases over its 90% quantile; and(0.9) measures the correlation when stock return increases over its 90% quantile and bond return decreases below its 10% quantile. For the other three groups, the exceeding correlations at the 10% quantiles $\tilde{\rho}_{\cdot,\cdot,t}$ (0.1) (or the 90% quantile $\tilde{\rho}_{\cdot,\cdot,t}$ (0.9)) calculate the correlation of both crashing below their 10% quantiles (and booming over their 90% quantiles).

Figure 3 (a) shows average bivariate correlations across 990 (=C^2_{45}) pairs of stocks. The correlations within the stock sector described by C^{skt} and C^{dskt} are similar. These stock ETFs are highly correlated, and the correlations are driven up further during the 2008 subprime crisis and the 2011 European debt crisis. This is evidence of financial contagion in stock sectors, as documented in Caporale, Cipollini, and Spagnolo (2005), Rodriguez (2007), and Kallberg and Pasquariello (2008), among others.

Figure 4. Excess Correlation for 45 Stocks, Bond, Oil, Exchange Rate, and Gold

The figures plot the exceeding correlations (at 10% and 90% quantiles) based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt}. Panel (a) is the average bivariate exceeding correlations across 990 pairs of stocks, panel (b) is the average exceeding correlations between each stock and bond across 45 pairs, panel (c) is the average exceeding correlations between each stock and oil across 45 pairs, and panel (d) is the exceeding correlations of foreign exchange and gold. The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.

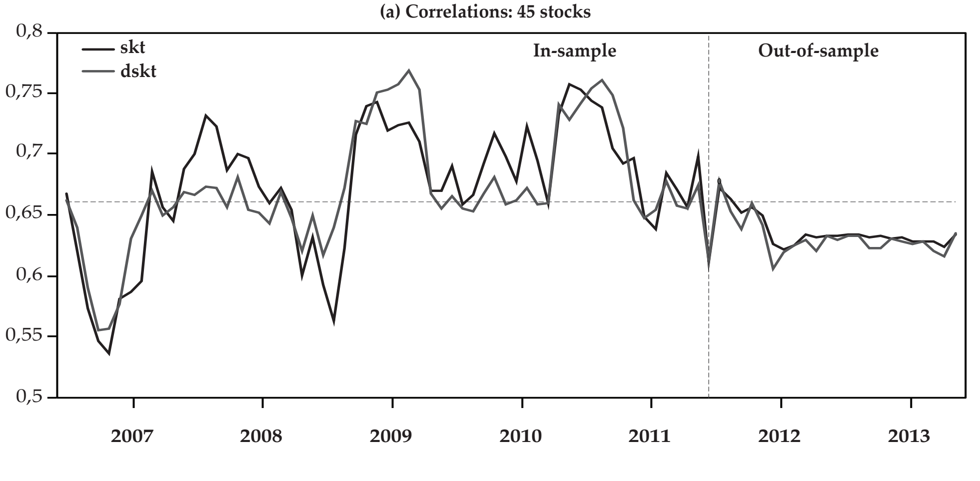


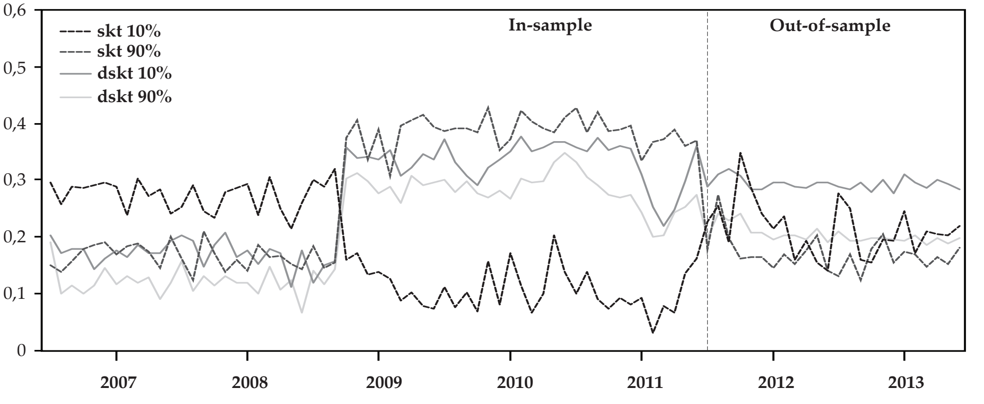
Figure 4 (a) reveals the various dependence patterns captured by C^{skt} and C^{dskt} via average bivariate exceeding correlations across 990 pairs of stock. From C^{skt} with $\bar{\gamma} = 0.43 > 0$, $\tilde{\rho}_{STK}$ (0.9) is always greater than $\tilde{\rho}_{STK}$ (0.1), implying these stocks are more likely to boom together than crash together throughout the sample interval. However, from C^{dskt}, $\tilde{\rho}_{STK}$ (0.9) > $\tilde{\rho}_{STK}$ (0.1) during the normal period due to positive skewness vector, and $\tilde{\rho}_{STK}$ (0.1) > $\tilde{\rho}_{STK}$ (0.9) due to negative skewness vector. This means that the dependence patterns within stock sectors change over time, which coincides with the results of Okimoto (2008), Guegan and Zhang (2010), and Elkamhi and Stefanova (2015). Hence, the dynamic specification of t enables us to distinguish the different dependence patterns in times of crisis and in normal periods.

Figure 4. Excess Correlation for 45 Stocks, Bond, Oil, Exchange Rate, and Gold (contd.)

The figures plot the exceeding correlations (at 10% and 90% quantiles) based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt}. Panel (a) is the average bivariate exceeding correlations across 990 pairs of stocks, panel (b) is the average exceeding correlations between each stock and bond across 45 pairs, panel (c) is the average exceeding correlations between each stock and oil across 45 pairs, and panel (d) is the exceeding correlations of foreign exchange and gold. The in-sample part is estimated from the sample of July 24, 2006 - June 30, 2011. The out-of-sample part is the one-step-ahead forecast with a rolling window of the past 1245 daily observations on each day of July 1, 2011 - June 28, 2013.

(b) Exceedance correlations: stock and bond

(c) Exceedance correlations: stock and oil



(d) Exceedance correlations: foreign exchange and gold

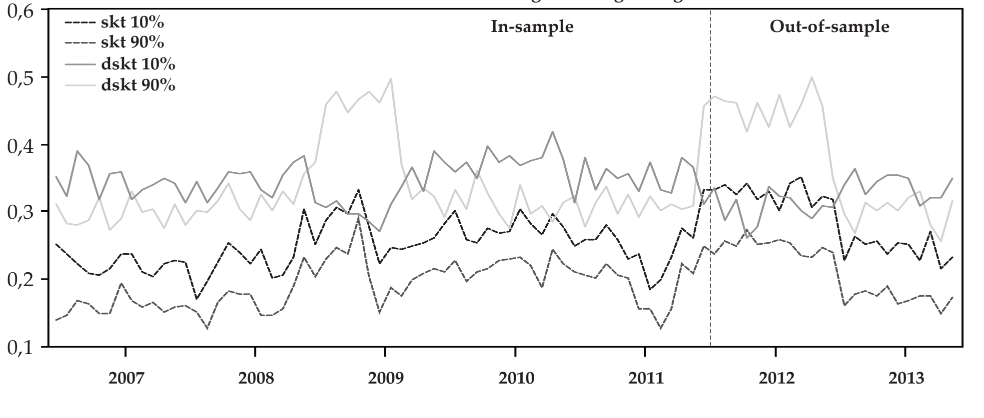


Figure 3 (b) provides the average correlations between each stock ETF and bonds across 45 pairs. The negative stock–bond correlation drops dramatically to less than -0.30 in the subprime crisis (September 2008–December 2008) and the European debt crisis (April 2010–November 2010).

Figure 4 (b) plots the exceeding correlations to demonstrate the difference in dependence patterns described by C^{skt} and C^{dskt}. C^{skt} captures only one dependence pattern that $|\tilde{\rho}_{*STK,AGG}(0.9)| > |\tilde{\rho}_{*STK,AGG}(0.1)|$, indicating that the dependence of rising stocks and declining bonds is stronger than the dependence of declining stocks and rising bonds. C^{dskt}, in contrast, depicts changing dependence patterns. In tranquil periods, $\tilde{\rho}_{*STK,AGG}(0.9)$ is close to $\tilde{\rho}_{*STK,AGG}(0.1)$ as $\gamma_{SKT,t} > 0$ and $\gamma_{AGG,t} < 0$ (see Figure 2 (a) and (b)). But in times of crisis, $|\tilde{\rho}_{*STK,AGG}(0.9)| < |\tilde{\rho}_{*STK,AGG}(0.1)|$ as both $\gamma_{SKT,t}$ and $\gamma_{AGG,t}$ become negative, indicating that the tendency for stock prices to decrease and for bond prices to increase is much stronger than the reverse tendency. The plunge in $\tilde{\rho}_{*STK,AGG}(0.1)$ is evidence of the "flight-to-safety" phenomenon during crises, also documented in Chan, Treepongkaruna, Brooks, and Gray (2011), Wu and Liang (2011), and Wu and Lin (2014). Overall, incorporation of the dynamic skewness vector makes C^{dskt} more flexible in describing the stock–bond dependence than C^{skt}.

In Figure 3 (c), we plot the average stock–oil correlation between each stock ETF and oil across 45 pairs. The stock–oil correlation is negative (-0.10) before the 2008 subprime crisis and becomes positive (0.30) afterwards. This implies to investors that the diversification benefits of stock and oil diminish after the subprime crisis.

In Figure 3 (c), we easily differentiate between C^{skt} and C^{dskt} by their exceeding correlations. From C^{skt}, $\tilde{\rho}_{STK,USO}(0.9)$ always exceeds $\tilde{\rho}_{STK,USO}(0.1)$ as $\bar{\gamma} > 0$. However, from C^{dskt}, due to the dynamic specification of the skewness vector, the stock–oil dependence exhibits a two-stage feature. Before the subprime crisis $\tilde{\rho}_{STK,USO}(0.9) > \tilde{\rho}_{STK,USO}(0.1)$, as $\gamma_{SKT,t}$ and $\gamma_{USO,t}$ are positive (see Figure 2 (a) and (e)); after the crisis, $\tilde{\rho}_{STK,USO}(0.9) < \tilde{\rho}_{STK,USO}(0.1)$, as $\gamma_{USO,t}$ and $\gamma_{SKT,t}$ drop below zero during the two crises. This means that stocks and oil are more likely to crash together than boom together after a crisis, again confirming the absence of diversification opportunities. Although the two-stage feature in correlation is also found in Chan et al. (2011) and Mollick and Assefa (2013), the two-stage feature in dependence patterns illustrated by exceeding correlations is rarely investigated. In short, our analysis of stock–oil dependence provides additional supporting evidence that C^{dskt} is more flexible than C^{skt} in describing multivariate dependence.

In Figure 3 (d), foreign exchange (EURO/US dollar) and gold (in dollars) become more positively dependent during the 2008 subprime crisis. For investors, gold no longer behaves as a safe haven during the crisis period, as stated in Sari, Hammoudeh and Soytas (2010), Joy (2011), and Pukthuanthong and Roll (2011).

In Figure 4 (d), C^{skt} and C^{dskt} can be easily distinguished by their exceeding correlations. Based on C^{skt} with $\bar{\gamma} > 0$, we may conclude that foreign exchange and gold stick to one dependence pattern, that $\tilde{\rho}_{FXE,GSG}(0.9) > \tilde{\rho}_{FXE,GSG}(0.1)$. The two ETFs are more dependent when booming simultaneously than crashing simultaneously. But this is not the case for C^{dskt} with dynamic skewness vector. During the 2008 subprime crisis, $\tilde{\rho}_{FXE,GSG}(0.9)$ is shown to be lower than $\tilde{\rho}_{FXE,GSG}(0.1)$ as both $\gamma_{FXE,t}$ and $\gamma_{GSG,t}$ drop sharply below zero (see Figure 2 (c) and (d)). This finding implies a higher likelihood of a depreciating Euro and a falling gold price following the subprime crisis. Investors managing portfolio risk should avoid holding both these assets for about six months. Again, this investigation of foreign exchange–gold dependence suggests the higher flexibility of C^{dskt} over C^{skt} in modeling multivariate dependence.

In summary, this section applies dynamic skewed t copula C^{dskt} to study the dependence of 50 ETF returns. The C^{dskt} model with dynamic skewness vector enables us to model multivariate dependence more flexibly and parsimoniously than existing copulas. We conclude that the 50 ETFs exhibit changing dependence patterns rather than only one dependence pattern throughout the sample interval.

V. OUT-OF-SAMPLE PERFORMANCE

This section investigates out-of-sample performance of the dynamic skewed t copula in the following two respects. First, we statistically analyze the predictive ability of our model C^{dskt}. Second, we evaluate the economic value of following the C^{dskt}-based strategy in an asset allocation problem.

A rolling sample method is utilized in the out-of-sample analysis. The 1245 observations from July 24, 2006 to June 30, 2011 are used for the in-sample estimation, and the 501 observations from July 1, 2011 to June 28, 2013 are left for the out-of-sample forecast. For each day, we re-estimate the model using the past 1245 observations. This procedure is repeated 501 times, so that 501 one-day-ahead joint distribution forecasts are produced.

To assess the predictive accuracy of our C^{dskt} model, we compare the predicted log-likelihood values of the five copulas mentioned in Section 4 and construct the Hansen (2005) superior predictive ability (SPA) test.

Table 4.
Predictive Ability Comparison of Copulas over the Out-of-Sample Period

This table reports the average of predictive log-likelihood (OOS) and its standard deviation (std) over the sample period of 2011/07/01- 2013/06/28. The last row shows the test statistic of Hansen's (2005) superior predictive ability and its p-value.

	Gaussian	t	skt	mskt	dskt
OOS	-161.6246	-162.8618	-163.1109	-161.8134	-159.3144
std	10.9788	12.3610	12.6055	13.3144	9.8596
SPA test stastic	3.3499		p-value	0.3500	

Table 4 reports the out-of-sample average of predicted log-likelihood of each copula (OOS) and its standard deviation (std), as in Lee and Long (2009). A model is expected to be closer to its true state if OOS is larger. C^{dskt} is shown to have the highest predicted log-likelihood and the lowest standard deviation. We conclude that, among the five copulas, C^{dskt} has the highest out-of-sample predictive ability in modeling the dependence of the 50 ETF returns.

Further, the Hansen (2005) SPA test gives us statistical justification for the outperformance of C^{dskt}. The null hypothesis is that the benchmark model is not inferior to any of the alternative models. Here, our dynamic skewed t copula C^{dskt} is set as the benchmark, while the other four copulas are regarded as alternatives. The SPA test statistic is 3.34 with p-value 0.35. Since we fail to reject the null at the

1% significance level, it can be inferred that C^{dskt} performs at least as well as the other copulas considered.

To explore the economic value of modeling dynamic multivariate dependence, we then consider the optimization problem of an investor allocating wealth among the 50 ETFs. Our hypothetical investor is characterized by a constant relative risk aversion utility, as in Patton (2004), and Wu and Lin (2014). At each period t, the investor solves the following optimization problem based on the one-step-ahead forecast to predict the portfolio weight at t +1:

$$\omega^{*}t+1=argmaxEt[U(rp,\ t+1;\eta)]$$

where $\omega_{t+1} = (\omega_{1,t+1}, \cdots, \omega_{N-1,t+1}, 1-\omega_{1,t+1} - \cdots - \omega_{N-1,t+1})$ are the weights on the N (=50) ETFs, and they can be negative without short selling constraints. $r_{p,t+1}$ is portfolio return, $r_{p,t+1} = \omega'_{t+1} r_{t+1}$, and $r_{t+1} = (r_{1,t+1}, \cdots, r_{N,t+1})'$ is the return vector of N ETFs at t+1. η is the degree of relative risk aversion and takes three levels: η = 1, 5, 10. The rolling window procedure is described at the beginning of this section.

Table 5 shows portfolio performance based on five copula models. We calculate each portfolio's annualized return (Mean), standard deviation (SD), terminal wealth value, 5% value-at-risk (VaR) and 5% expected shortfall (ES). Besides, following Wu and Liang (2011) and Wu and Lin (2014), we compute the performance fee (PF) that an investor is willing to pay to switch from another copula-based strategy to our dynamic skewed t copula-based strategy. A positive PF means the C^{dskt}-based strategy is better than the alternative strategy using another copula.

Table 5.
Portfolio Performance Comparison of Copula over the Out-of-Sample Period

This table summarizes the copula-based portfolio performance over the sample period of 2011/07/01-2013/06/28. η represents the coefficient of relative risk aversion. The sample mean, standard deviation (SD), 5% value-at-risk (VaR) and 5% expected shortfall have been annualized. PF denotes the performance fee an investor will pay if she switches from other copula-based strategy to the dynamic skewed t copula C^{dskt} based strategy. A positive PF means the portfolio based on C^{dskt} performs better than the portfolio based on the other copula.

		Gaussian	*t*	skt	mskt	dskt
η = 1	Mean (%)	7.6117	6.3292	2.7480	9.3198	12.2894
	SD (%)	8.1428	5.9066	3.3932	10.1465	10.2767
	Terminal wealth	115.8027	113.0589	105.5715	119.5081	126.0890
	VaR (5%)	7.0741	8.5426	9.5672	7.0266	4.8034
	ES (5%)	10.1067	13.6605	13.9828	9.2268	7.8926
	PF	4.1039	4.9972	7.6096	3.2681	
η = 5	Mean (%)	3.417	2.5577	1.1831	5.1931	5.6061
	SD (%)	4.0111	3.3487	2.4675	5.0202	5.1792
	Terminal wealth	106.9508	105.1809	102.3802	110.656	111.5264
	VaR (5%)	5.1879	7.0258	8.7943	6.4351	3.4489
	ES (5%)	7.848	9.6845	14.7601	9.1713	5.0896
	PF	1.9500	2.6848	3.2373	0.4421	
η = 10	Mean (%)	1.0821	1.6425	-1.3837	0.5349	2.5969
	SD (%)	2.4613	3.4825	6.3294	4.9361	3.7858
	Terminal wealth	102.1760	103.3121	97.2518	101.0727	105.2612
	VaR (5%)	4.2958	4.9818	7.5480	6.0646	3.1901
	ES (5%)	6.1846	6.7866	13.5201	9.2427	4.6196
	PF	1.1105	0.9195	6.9027	3.2612	

In Table 5, among the five strategies, the strategy using C^{dskt} performs best, followed by the strategies using $C^{Gaussian}$, C^{t} and C^{mskt}, while C^{skt} performs worst. This ranking is based on the values of performance fee and is unaffected by the levels of investor risk aversion.

An investor who disregards changing dependence patterns incurs losses when modeling high dimensional financial returns. For example, the performance fee from C^{mskt} to C^{dskt} for an investor with $\eta = 10$ is 3.26 cents per dollar. The gains from considering changing dependence patterns are also revealed in Okimoto (2008) and Elkamhi and Stefanova (2015). It is inferred that higher-dimensional financial returns usually exhibit changing dependence patterns, and rarely do they follow just one dependence pattern over time. Hence, employing a flexible dynamic model to accommodate this changing dependence can make investors better off.

Further, portfolio performance is sensitive to the correctness of dependence characterization. If the dependence structure is predicted based on an incorrect copula, such as C^{skt}, the corresponding portfolio will perform even worse than portfolios that simply use correlation-based models like $C^{Gaussian}$ and C^{t}.

Why do strategies using C^{skt} and C^{dskt} perform so differently? Figure 5 investigates the portfolio weights resulting from C^{skt} and C^{dskt}. It is evident that the two types of investor have different opinions mainly about their weights on stocks, bonds, and foreign exchange.

Figure 5. Comparing Portfolio Weights for C^{skt} and C^{dskt}

The figures plot the portfolio weights on 50 ETFs based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt} over the sample period of July 1, 2011 - June 28, 2013. Panel (a) is the average weights on 45 stock ETFs, panel (b)-(f) are the average weights on bond, foreign exchange, gold, oil and real estate.

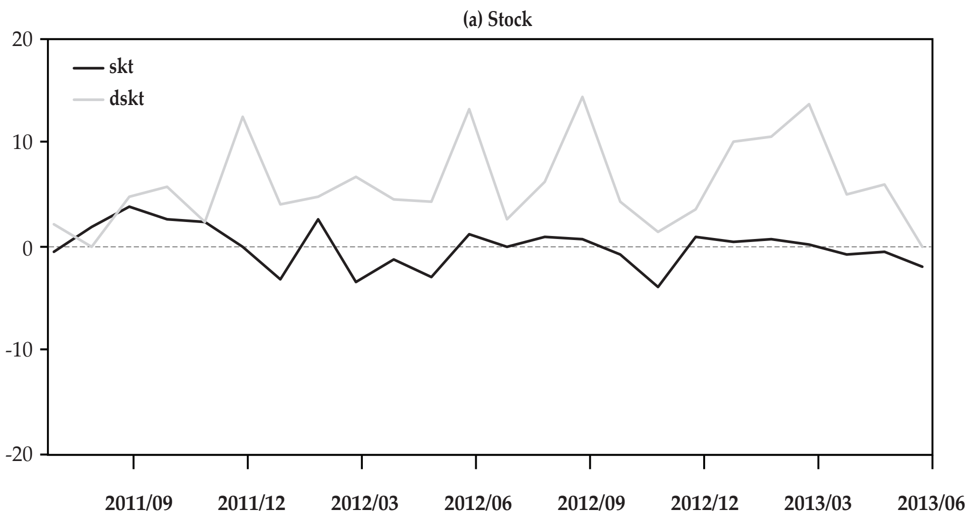
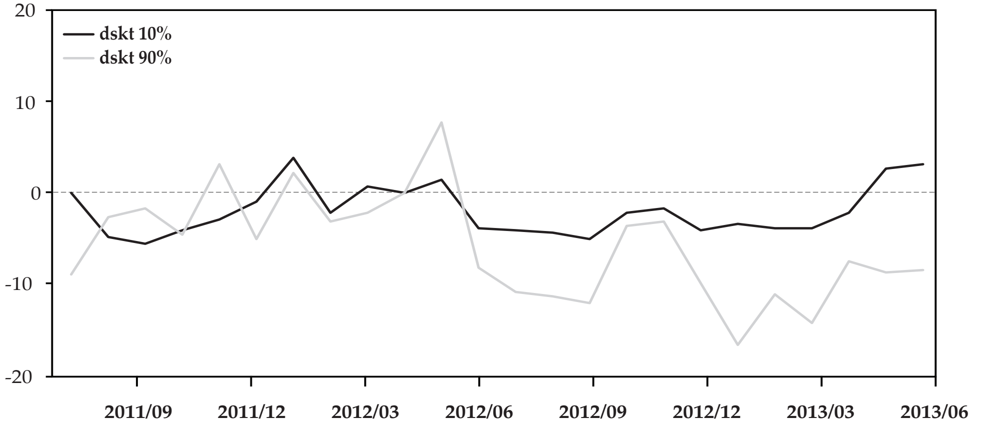


Figure 5 (a) plots the average weights on 45 stock ETFs and illustrates that the investor following C^{dskt} buys more stocks than the investor following C^{skt}. This is because C^{dskt} predicts higher 90% exceeding correlations than C^{skt} (see Figure 5 (a)). In other words, the C^{dskt}-based investor is more likely to believe that the stocks will increase together, so he will hold more stocks than the C^{skt}-based investor.

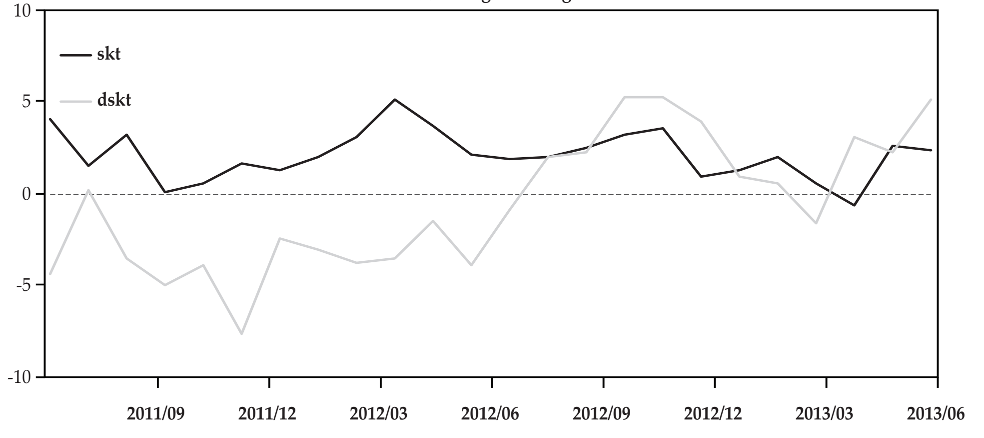
Figure 5. Comparing Portfolio Weights for C^{skt} and C^{dskt} (contd.)

The figures plot the portfolio weights on 50 ETFs based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt} over the sample period of July 1, 2011 - June 28, 2013. Panel (a) is the average weights on 45 stock ETFs, panel (b)-(f) are the average weights on bond, foreign exchange, gold, oil and real estate.

(b) Bond



(c) Foreign exchange



(d) Gold

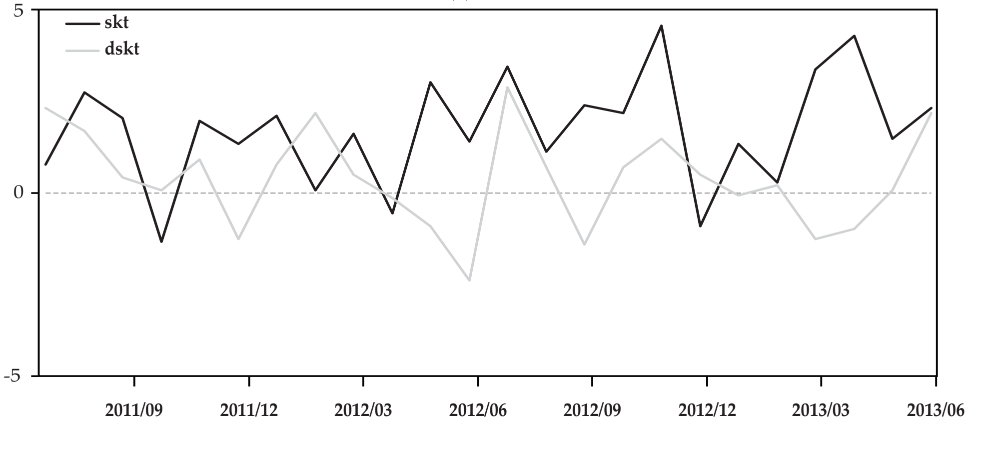
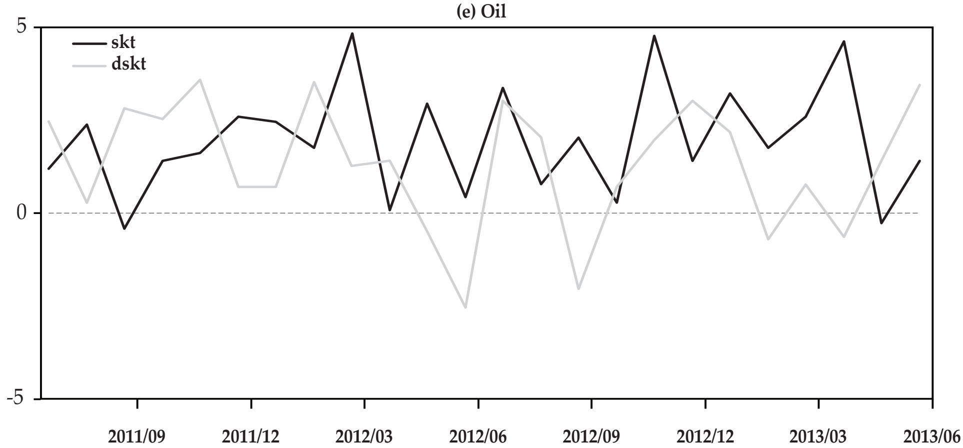


Figure 5. Comparing Portfolio Weights for C^{skt} and C^{dskt} (contd.)

The figures plot the portfolio weights on 50 ETFs based on the skewed t copula model C^{skt} and the dynamic skewed copula model C^{dskt} over the sample period of July 1, 2011 - June 28, 2013. Panel (a) is the average weights on 45 stock ETFs, panel (b)-(f) are the average weights on bond, foreign exchange, gold, oil and real estate.



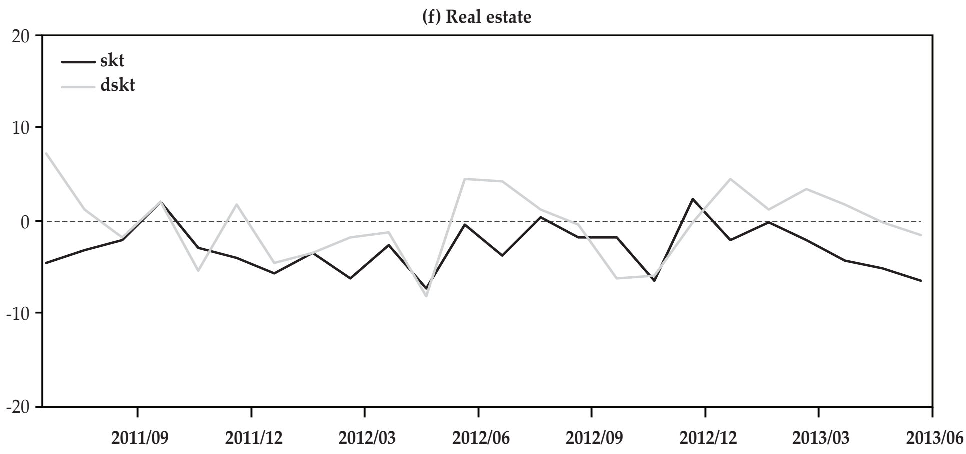


Figure 5 (b) shows that the investor using C^{dskt} shorts more bond than the investor using C^{skt}. The reason for this is similar: the 90% exceeding correlations from C^{dskt} are higher (in absolute value) in those from C^{skt} (see Figure 4 (b)). From the perspective of the C^{dskt}-based investor, bond price has a higher probability of decreasing when stock price increases, hence this investor tries to sell more bonds than the C^{skt} counter-partner.

Figure 5 (c) indicates the investor with C^{skt} mostly longs foreign exchange, while the investor with C^{dskt} shorts it before June 2012 and longs it afterwards. This difference in positions is caused by their forecasts of skewness parameters. C^{skt} predicts a positive skewness scalar for both stock and foreign exchange (see Figure 2 (a) and (c)), and the investor buys both of them believing the two assets will increase together. In contrast, C^{dskt} also predicts positive C^{skt}, but the investor believes γ_{FXE} is negative during the period July 2011 to June 2012 and becomes positive afterwards. In this investor's opinion, stocks and foreign exchange before June 2012 are not as closely correlated as they are after June 2012, so the investor sells foreign exchange before June 2012 and then buys it again.

This result is obtained based on a longer in-sample period and a relatively shorter out-of-sample period. As a robustness check, we follow Narayan and Bannigidadmath (2015) and split our sample from June 24, 2006 to June 28, 2013 into a 50:50 in-sample and out-of-sample ratio. The in-sample period is now June 24, 2006 to December 31, 2009, and the out-of-sample period is January 4, 2010 to June 28, 2013. Table 6 reports in-sample estimates of copula models, Table 7 shows out-of-sample predictive ability test results, and Table 8 shows out-of-sample portfolio performance.[8]

Table 6.
Copula Model Estimates Over the In-Sample Period: 2006/06/24 – 2009/12/31

This table reports the estimates of copula models over the sample period of 2006/06/24 – 2009/12/31. Standard errors are in the brackets below the parameters. The ***, ** and * denote statistical significance levels at 1%, 5%, and 10%, respectively. logL denotes the log-likelihood value of each model, including both copula and marginal models.

	Gaussian		*t*		**skt**		**mskt**		**dskt**
α_1	0.0079***	α_1	0.0052***	α_1	0.0040*	α_1	0.0142***	α_1	0.0127***
	(0.0013)		(0.0009)		(0.0022)		(0.0035)		(0.0032)
α_2	0.9254***	α_2	0.9073***	α_2	0.8007**	α_2	0.8621**	α_2	0.8304***
	(0.0161)		(0.0244)		(0.3300)		(0.5809)		(0.0565)
		v	13.2215***	v	13.0034***	v	14.3011***	v	15.0218***
			(0.7221)		(0.1193)		(0.8183)		(0.4765)
				$\bar{\gamma}$	0.0064***	γ_{STK}	0.4872***	$\beta 1$	-0.7094**
					(0.0230)		(0.0205)		(0.2992)
						γ_{AGG}	-0.2425***	$\beta 2$	0.7561**
							(0.1985)		(0.3699)
						γ_{FXE}	0.0472***		
							(0.2455)		
						γ_{GSG}	0.2473***		
							(0.2473)		
						γ_{USO}	0.2848**		
							(0.1293)		
						γ_{RWR}	0.4835***		
							(0.0428)		
logL (x 10^2)	-1.8413		-1.8316		-1.7211		-1.6355		-1.6053
AIC (x 10^2)	3.7226		3.7233		3.5222		3.4510		3.3107
BIC (x 10^2)	3.7414		3.7516		3.5600		3.5359		3.3579

8 To save space, we skip the in-sample estimates of 50 marginal modes; they are available upon request.

Overall, the results are robust when the sample is split 50:50. In Table 6, the dynamic skewed t copula C^{dskt} provides the best in-sample goodness-of-fit among the five models with the lowest AIC and SIC. In Table 7, the dynamic skewed t copula C^{dskt} has the highest predictive ability and we fail to reject the SPA test, though predictive ability is lowered compared with the result shown in Table 4. In Table 8, the out-of-sample portfolio performance of the C^{dskt}-based strategy is still best, as it has the highest average return (Mean) and terminal wealth, and the lowest standard deviation, VaR, and expected shortfall. The performance fees of switching from other copula models to C^{dskt} are positive regardless of the risk aversion levels. The only difference is that the portfolios in this scenario with longer out-of-sample period (Table 8) exhibit higher returns but higher risks (such as VaR and ES) than the portfolios in the original scenario (Table 5).

Table 7.
Predictive Ability Comparison of Copulas Over The Out-of-Sample Period: 2010/01.04-2013/06/28

This table reports the average of predictive log-likelihood (OOS) and its standard deviation (std) over the sample period of 2010/01/04- 2013/06/28. The last row shows the test statistic of Hansen's (2005) superior predictive ability and its *p*-value.

	Gaussian	*t*	skt	mskt	dskt
OOS	-165.8489	-187.2045	-202.6078	-179.9992	-162.7837
std	10.4253	8.9757	7.6717	7.6398	7.3134
SPA test stastic	4.3964		*p*-value	0.5260	

Table 8.
Portfolio Performance Comparison of Copula Over the Out-of-Sample Period: 2010/01/04-2013/06/28

This table shows the copula-based portfolio performance over the sample period of 2010/01/04-2013/06/28. η represents the coefficient of relative risk aversion. The sample mean, standard deviation (SD), 5% value-at-risk (VaR) and 5% expected shortfall (ES) have been annualized. PF denotes the performance fee an investor will pay if he switches from other copula-based strategy to the dynamic skewed *t* copula (dskt) based strategy. A positive PF means the portfolio based on dskt performs better than the portfolio based on the other copula.

		Gaussian	*t*	skt	mskt	dskt
η =1	Mean (%)	5.6245	5.1260	2.8879	5.2069	5.9158
	SD (%)	5.2256	4.2342	4.1870	4.5801	4.1862
	Terminal wealth	121.0039	119.0257	110.4277	119.3453	122.1703
	VaR (95%)	17.6167	17.5622	21.1368	24.7608	23.7524
	ES (95%)	25.4321	24.0339	30.4241	28.3839	24.8907
	PF	0.3566	0.9695	3.1345	0.8698	
η =5	Mean (%)	4.6343	4.6958	2.3312	4.3844	5.3373
	SD (%)	5.2682	3.8872	2.8345	3.8955	4.5842
	Terminal wealth	117.0972	117.3375	108.3600	116.1259	119.8612
	VaR (95%)	14.5856	14.0884	16.7841	17.6962	14.5177
	ES (95%)	20.4882	20.8635	24.6501	21.6531	19.8022
	PF	0.6494	0.9195	1.8933	0.9272	
η =10	Mean (%)	3.2174	3.2886	1.9267	2.8530	4.4226
	SD (%)	3.1812	3.1726	1.8923	3.4111	4.0847
	Terminal wealth	111.6649	111.9336	106.8752	110.2974	116.2739
	VaR (95%)	11.2303	12.2545	14.5709	13.1395	11.8310
	ES (95%)	17.3096	17.1338	20.0210	21.9134	18.5828
	PF	0.8859	0.8360	2.0405	1.3275	

In summary, the predictive ability tests and portfolio optimization problem in this section demonstrate that our dynamic skewed t copula outperforms existing copulas, such as those of Christoffersen et al. (2012) and Gonzalez-Pedraz et al. (2015). Investors who account for changing multivariate dependence patterns are able to forecast the joint distribution of 50 ETF returns more precisely and become better off when allocating their wealth across these assets. This illustrates the importance of considering changing dependence patterns in modeling high dimensional financial returns from an investor point of view.

VI. CONCLUSION

Developing copula models to describe high dimensional multivariate dependence is of particular interest to investors who allocate their wealth among a large number of assets. Existing skewed t copulas with static skewness parameters are unable to capture the time variation in dependence patterns. This paper complements the literature by modeling the dynamics of high dimensional multivariate dependence in a flexible yet parsimonious way. We extend the copula in Christoffersen et al. (2012) and Gonzalez-Pedraz et al. (2015) and propose a dynamic skewed t copula. The new model not only allows each variable to have its own skewness parameter, but also incorporates an autoregressive mechanism into the evolution of the skewness vector.

Applying our dynamic skewed *t* copula C^{dskt} to 50 ETF returns, we find that C^{dskt} has better in-sample and out-of-sample performance than other copulas. In the in-sample analysis, we find that the dependence of these ETFs during crisis periods differs from their dependence during tranquil periods, and this feature can be captured only by the dynamic specification of skewness vector in our model. This indicates that the diversification benefit of most assets (except bonds) is limited during crises. The out-of-sample analysis shows the outperformance of C^{dskt} over existing copulas based on the predicted log-likelihood values and the portfolio optimization problem. The results documented in this study imply that a large number of financial assets tend to have time-varying dependence patterns rather than static ones. It is thus inappropriate to stick to one dependence structure when investors are modeling high dimensional financial returns.

This paper leaves several topics for further research, including how to increase the estimation efficiency of high dimensional copulas. Future research should also examine the dependence of financial returns in emerging markets with greater fluctuations. These results from the dynamic skewed t copula may be of interest to policy makers and market participants alike.

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